A skew-symmetric property of rigid-body systems¹

Kuang-Yow Lian^{a,*}, Li-Sheng Wang^b, Li-Chen Fu^c

^a Department of Electrical Engineering, Chung-Yuan Christian University, Chung-Li, Taiwan

^b Institute of Applied Mechanics, National Taiwan University, Taipei, Taiwan

^c Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan

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Abstract

The configuration space for the attitude of a vehicle can be modeled as $SO(3)$, namely the rotation matrix group. This work investigates the globally valid dynamics of natural Lagrangian systems and gyroscopic systems with configuration space including $SO(3)$. The dynamics is derived by using the global representations of jet bundles of $SO(3)$. A skew-symmetric property associated with the systems can be then established. Such property can be used in many applications such as the adaptive controller design. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In modeling space robots, satellites, and spacecraft systems, it is necessary to describe the attitude or the orientation of a rigid body. Mathematically, the attitude of a rigid body could be modeled by (3×3) rotation matrices, which as a whole form the *Special Orthogonal Group* $SO(3)$. Conventional studies that deal with attitude dynamics and control typically use local coordinates, e.g. Euler angles, which inevitably encounter singularity problems. On the other hand, some studies adopt the representation of unit quaternions for modeling system dynamics. In this work, the global representations of the jet bundle of $SO(3)$, i.e., coordinate-free representations, are adopted to derive the attitude dynamics and to perform the analysis. A concise formulation of Euler–Lagrange equation on $TSO(3)$ is obtained, whereby the equations of motion for general mechanical systems with configuration space $SO(3)^m \times \mathbb{R}^n$ can be thus derived. The merit of this approach is manifest after a skew-symmetric property pertaining to the systems being established. Such property for the mechanical systems expressed in $\mathbb{R}^n$ has been extensively exploited in the controller design such as [10]. It is also noted for some specific rigid-body systems in [3, 4]. However, here this property is rigorously proven to be valid for general mechanical systems with configuration space $SO(3)^m \times \mathbb{R}^n$. This result facilitates the *globally valid* controller design for mechanical systems with configuration space including $SO(3)$ (cf. [5]).

* Corresponding author.

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2. Lagrangian mechanics in invariant form

Some basic terms in Lagrangian mechanics are described in this section which naturally lead to the Euler–Lagrange equations on the configuration space $SO(3)$. First we describe the Lagrange–d’Alembert principle for Lagrangian systems in an invariant form.

Let Q be a smooth, n -dimensional manifold with local coordinates $x = (x_1, x_2, \dots, x_n)$, TQ be the tangent bundle on Q with local coordinates $(x, v) = (x, v_1, v_2, \dots, v_n)$, and $\pi_Q: TQ \rightarrow Q$ be the canonical projection defined by $\pi_Q(x, v) \triangleq x$. Let TTQ be the second tangent bundle on Q and $T\pi_Q: TTQ \rightarrow TQ$ be the tangent map of π_Q . Letting $\omega (= \sum_{i=1}^n \omega_i dx_i$ in local coordinates) be a 1-form on Q , a function $\iota\omega$ on TQ can be naturally defined as $\iota\omega(x, v) = \sum_{i=1}^n \omega_i(x) v_i$, in local coordinates (cf. [15]). Denote the set of smooth functions on the manifold Q as $C^\infty(Q)$, the set of vector fields as $\mathcal{X}(Q)$, and the set of k -forms as $\mathfrak{w}^k(Q)$. We call $f^\vee \triangleq f \circ \pi \in C^\infty(TQ)$ the *vertical lift* of the function $f \in C^\infty(Q)$.

Definition 1. The *vertical vector field* $X^\vee$ in $\mathcal{X}(TQ)$ is defined by $X^\vee f^\vee = 0$, $\forall f \in C^\infty(Q)$; the *principal vertical field* X^{pv} in $\mathcal{X}(TQ)$ is defined by $X^{pv}(\iota\omega) = \iota\omega$, $\forall \omega \in \mathfrak{w}^1(Q)$; the *horizontal 1-form* ω^h in $\mathfrak{w}^1(TQ)$ is defined by $\omega^h(X^\vee) = 0$, $\forall X^\vee$; the *special vector field* X^s in $\mathcal{X}(TQ)$ is defined by $T\pi_Q(X^s) = \pi_{TQ}(X^s)$; the map $\tau: \mathfrak{w}^1(TQ) \rightarrow \mathfrak{w}^1(TQ)$ is defined by $(\tau \cdot \omega) \cdot X^s = \omega \cdot X^{pv}$, $\forall X^s, X^{pv}, \omega \in \mathfrak{w}^1(TQ)$.

In local coordinates, we have the following properties.

Proposition 1. Let $X \in T_{(x,v)}TQ$ and $\omega \in T_{(x,v)}^*TQ$. (i) $X^\vee(x, v) = (0, w)$, for some $w \in \mathbb{R}^n$; (ii) $X^{pv}(x, v) = (0, v)$; (iii) $\omega^h(x, v) = (\alpha, 0)$, for some $\alpha \in \mathbb{R}^n$; (iv) $X^s(x, v) = (v, w)$, for some $w \in \mathbb{R}^n$; (v) $\tau_{(x,v)} \cdot (\alpha, \beta) = (\beta, 0)$.

Proof. For simplicity, we only prove (i) and the rest follows suit. Let

$$\sum_{i=1}^n u_i \frac{\partial}{\partial x_i} + w_i \frac{\partial}{\partial v_i}$$

be the coordinate expression of $X^\vee$. Then, from $X^\vee f^\vee = 0$, we have $\sum_{i=1}^n u_i \partial f / \partial x_i = 0$ for all $f \in C^\infty(Q)$, from which $u_i = 0$, $\forall i = 1, \dots, n$, i.e., $X^\vee(x, v) = (0, w)$. This completes the proof. $\square$

The correspondence between mechanical notions and the geometric objects can be found in [11]. Now, let the smooth manifold Q be the configuration space of a mechanical system and let a smooth function L on TQ be the *Lagrangian*. Define a horizontal 1-form by $\Theta_L \triangleq \tau \cdot dL$, where $dL: TQ \rightarrow T^*TQ$ is an exterior derivative of L . In local coordinates, $dL(x, v) = (D_1 L(x, v), D_2 L(x, v))$, where $D_i L(x, v)$ is the Fréchet derivative of L relative to the i th argument of L . Then, $\Theta_L(x, v) = (D_2 L(x, v), 0)$. The *Lagrangian force* on a special vector field X^s , denoted as $F_L(X^s) \in \mathfrak{w}^1(TQ)$, is defined through

$$F_L(X^s) \triangleq -d\Theta_L(X^s, \cdot) - d(dL(X^{pv})) + dL,$$

where $d\Theta_L$ is a 2-form. One can show that $F_L(X^s)$ is horizontal and its expression in local coordinates is, for $X^s(x, v) = (v, w)$,

$$F_L(X^s)(x, v)(u, w) = (-D_1 D_2 L(x, v) \cdot v - D_2 D_2 L(x, v) \cdot w + D_1 L(x, v)) \cdot u, \quad (1)$$

for any $(u, w) \in T_{(x,v)}TQ$, cf. [13]. In the above setting, the Lagrange–d’Alembert Principle [11] says that the special vector field X^s whose integral curves are the trajectories of motion in TQ satisfies

$$(F_L(X^s) + \omega_e) \cdot Z = 0, \quad \forall Z \in \mathcal{X}(TQ), \quad (2)$$

where ω_e is a horizontal 1-form on TQ denoting the exterior force. Let $Z(x, v) \triangleq (u, w)$ and $\omega_e(x, v) \triangleq (\alpha, 0)$. Substituting $F_L(X^s)$ in Eq. (1), we have the local expression of Eq. (2),

$$(-D_1 D_2 L(x, v) \cdot v - D_2 D_2 L(x, v) \cdot w + D_1 L(x, v) + \alpha) \cdot u = 0,$$

or, by letting $v = \dot{x}$ and $w = \dot{v}$,

$$\left(\frac{d}{dt} D_2 L(x, v) - D_1 L(x, v) \right) \cdot u = \alpha \cdot u. \quad (3)$$

For finite-dimensional case, Eq. (3) is, in fact, reduced to the classical Euler–Lagrange equation in vector form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \alpha. \quad (4)$$

In the following section, Eq. (2) will be used to derive the globally valid attitude dynamics.

3. Euler–Lagrange equation on TSO(3)

Let the operator “ $\wedge$ ” denote the natural isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3)$, the space of 3×3 skew-symmetric matrices, defined by

$$\widehat{w} \triangleq \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad \forall w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3.$$

The inverse map of $\wedge$ is denoted by “ $\vee$ ”, that is, $\vee: \mathfrak{so}(3) \rightarrow \mathbb{R}^3$; $\widehat{w} \mapsto w$. The attitude of a rigid body can be represented by the rotation matrix of a body frame relative to an inertial frame. All such rotation matrices form the *special orthogonal group* $SO(3)$. For arbitrary $B \in SO(3)$, one has the kinematic relationship $\dot{B} = B\widehat{\Omega}$, where the variable Ω corresponds to the body angular velocity [6]. This means that we could use $(B, B\widehat{\Omega})$ to represent the element of the tangent bundle on $SO(3)$, i.e., $TSO(3)$. Moreover, with respect to the trace pairing in $GL(n)$ (general linear group) defined by, $\langle M, N \rangle = \frac{1}{2} \text{tr}(M^T N)$, $M, N \in GL(n)$, the representations for $T_{(B, B\widehat{\Omega})} TSO(3)$ and $T_{(B, B\widehat{\Omega})}^* TSO(3)$ are, respectively, $(B\widehat{u}, B(\widehat{u}\widehat{\Omega} + \widehat{w}))$ and $(B(\widehat{b}\widehat{\Omega} + \widehat{a}), B\widehat{b})$ with $u, w, a, b \in \mathbb{R}^3$, cf. [6]. Restricting to $SO(3)$, the Lagrange–d’Alembert Principle in terms of the global representations is

$$d\Theta_L(X^s, Z) + d(dL(B, B\widehat{\Omega})(X^{pv}))(Z) - dL(B, B\widehat{\Omega})(Z) - \omega_e(Z) = 0, \quad (5)$$

where $X^s = (B\widehat{\Omega}, B(\widehat{\Omega}\widehat{\Omega} + \widehat{w}))_{(B, B\widehat{\Omega})}$, for arbitrary $w \in \mathbb{R}^3$, $X^{pv} = (0, B\widehat{\Omega})_{(B, B\widehat{\Omega})}$, and Z could be represented as $(B\widehat{u}_1, B(\widehat{u}_1\widehat{\Omega} + \widehat{w}_1))_{(B, B\widehat{\Omega})}$, for arbitrary $u_1, w_1 \in \mathbb{R}^3$. The exterior force, ω_e , on the phase variable $(B, B\widehat{\Omega})$ can be modeled as a horizontal 1-form on $T_{(B, B\widehat{\Omega})} SO(3)$: $(\alpha_e, 0) = (B\widehat{\alpha}, 0)$, $\alpha \in \mathbb{R}^3$. The terms of Eq. (5) can be derived as follows:

$$\begin{aligned} d\Theta_L(X^s, Z) &= (d(D_2 L(B, B\widehat{\Omega}), 0) \cdot X^s)(Z) - (d(D_2 L(B, B\widehat{\Omega}), 0) \cdot Z)(X^s) \\ &= \left\langle \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} D_2 L(B e^{\varepsilon \widehat{\Omega}}, B e^{\varepsilon \widehat{\Omega}}(\widehat{\Omega} + \varepsilon \widehat{w})), B\widehat{u}_1 \right\rangle - \left\langle \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} D_2 L(B e^{\varepsilon \widehat{u}_1}, B e^{\varepsilon \widehat{u}_1}(\widehat{\Omega} + \varepsilon \widehat{w}_1)), B\widehat{\Omega} \right\rangle, \\ d(dL(B, B\widehat{\Omega})(X^{pv}))(Z) &= d(D_2 L(B, B\widehat{\Omega}), B\widehat{\Omega})(Z) \end{aligned}$$

$$\begin{aligned}
&= \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \langle D_2 L(Be^{\varepsilon \hat{u}_1}, Be^{\varepsilon \hat{u}_1}(\hat{\Omega} + \hat{w}_1)), Be^{\varepsilon \hat{u}_1}(\hat{\Omega} + \hat{w}_1) \rangle \\
&= \left\langle \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} D_2 L(Be^{\varepsilon \hat{u}_1}, Be^{\varepsilon \hat{u}_1}(\hat{\Omega} + \varepsilon \hat{w}_1)), B\hat{\Omega} \right\rangle + \langle D_2 L(B, B\hat{\Omega}), B(\hat{u}_1 \hat{\Omega} + \hat{w}_1) \rangle, \\
&dL(B, B\hat{\Omega})(Z) \\
&= \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} L(Be^{\varepsilon \hat{u}_1}, Be^{\varepsilon \hat{u}_1}(\hat{\Omega} + \varepsilon \hat{w}_1)) \\
&= \langle D_1 L(B, B\hat{\Omega}), B\hat{u}_1 \rangle + \langle D_2 L(B, B\hat{\Omega}), B(\hat{u}_1 \hat{\Omega} + \hat{w}_1) \rangle,
\end{aligned}$$

and $\omega_e(Z) = \langle \alpha_e, B\hat{u}_1 \rangle$. Substituting the above expressions into Eq. (5), we obtain

$$\left\langle \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} D_2 L(Be^{\varepsilon \hat{\Omega}}, Be^{\varepsilon \hat{\Omega}}(\hat{\Omega} + \varepsilon \hat{w})) - D_1 L(B, B\hat{\Omega}) - \alpha_e, B\hat{u}_1 \right\rangle = 0. \quad (6)$$

With $w = \hat{\Omega}$ and the kinematic relationship, it can be checked that

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} D_2 L(Be^{\varepsilon \hat{\Omega}}, Be^{\varepsilon \hat{\Omega}}(\hat{\Omega} + \varepsilon \hat{w})) = \frac{d}{dt} D_2 L(B, B\hat{\Omega}).$$

Moreover, since B is nonsingular and u_1 is arbitrary, Eq. (6) leads directly to the Euler–Lagrange equation shown in the following theorem, cf. [13].

Theorem 1. On $\text{TSO}(3)$, let a mechanical system be described by a Lagrangian $L(B, B\hat{\Omega})$, where $B\hat{\Omega} \in T_B \text{SO}(3)$. The Lagrange–d’Alembert Principle in the invariant form [11] gives rise to the Euler–Lagrange equation,

$$\frac{d}{dt} D_2 L(B, B\hat{\Omega}) - D_1 L(B, B\hat{\Omega}) = \alpha_e, \quad (7)$$

where α_e is the exterior force, which can be treated as an element in $\mathfrak{w}^1(\text{TSO}(3))$.

However, it is typical in Lagrangian mechanics to express compactly the Lagrangian in terms of (B, Ω) , instead of $(B, B\hat{\Omega})$, i.e., $L(B, B\hat{\Omega}) = \tilde{L}(B, \Omega)$. In fact, coordinates (B, Ω) can be regarded as a representation of $\text{SO}(3) \times \mathbb{R}^3$ which is isomorphic to $\text{TSO}(3)$. Accordingly, directly applying Eq. (7) is inconvenient. It is thus desired to have dynamical equations in terms of the variables (B, Ω) . By definitions,

$$\begin{aligned}
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} L(Be^{\varepsilon \hat{u}}, Be^{\varepsilon \hat{u}}(\hat{\Omega} + \varepsilon \hat{w})) &\triangleq \langle (D_1 L, D_2 L), (B\hat{u}, B(\hat{u}\hat{\Omega} + \hat{w})) \rangle \\
&\triangleq \langle (B(\hat{b}\hat{\Omega} + \hat{a}), B\hat{b}), (B\hat{u}, B(\hat{u}\hat{\Omega} + \hat{w})) \rangle \\
&= a^\top u + b^\top w,
\end{aligned} \quad (8)$$

where $u, w, a, b \in \mathbb{R}^3$. On the other hand,

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} L(Be^{\varepsilon \hat{u}}, Be^{\varepsilon \hat{u}}(\hat{\Omega} + \varepsilon \hat{w})) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \tilde{L}(Be^{\varepsilon \hat{u}}, (\hat{\Omega} + \varepsilon \hat{w})) = \langle D_B \tilde{L}, B\hat{u} \rangle + \left(\frac{\partial \tilde{L}}{\partial \Omega} \right)^\top w, \quad (9)$$

where $D_B \tilde{L}$ denotes the Fréchet derivative of $\tilde{L}$ relative to B . Note that $D_1 L(B, B\hat{\Omega})$ belongs to the first component of an element in $T_{(B, B\hat{\Omega})}^* \text{TSO}(3)$, whereas $D_B \tilde{L}(B, \Omega)$ belongs to an element in $T_B^* \text{SO}(3)$. Comparing Eq. (8) with Eq. (9), we have the following relations: $\hat{a} = B^\top D_B \tilde{L}$ and $b = \partial \tilde{L} / \partial \Omega$, which imply that

$$D_1 L = B \left(\frac{\partial \tilde{L}}{\partial \Omega} \hat{\Omega} + B^\top D_B \tilde{L} \right), \quad D_2 L = B \frac{\partial \tilde{L}}{\partial \Omega}.$$

Substitution of the above equations in the Euler–Lagrange equation (7) yields

$$\begin{aligned} 0 &= \frac{d}{dt} D_2 L(B, \widehat{B\Omega}) - D_1 L(B, \widehat{B\Omega}) - B\hat{\alpha} \\ &= B \frac{d}{dt} \frac{\partial \bar{L}}{\partial \Omega} + B\widehat{\Omega} \frac{\partial \bar{L}}{\partial \Omega} - B \frac{\partial \bar{L}}{\partial \Omega} \widehat{\Omega} - BB^\top D_B \bar{L} - B\hat{\alpha} \\ &= B \left(\frac{d}{dt} \frac{\partial \bar{L}}{\partial \Omega} + \left(\Omega \times \frac{\partial \bar{L}}{\partial \Omega} \right)^\wedge - B^\top D_B \bar{L} - \hat{\alpha} \right), \end{aligned}$$

which then lead to the following theorem.

Theorem 2. On $\text{TSO}(3)$, let a mechanical system be described by a Lagrangian $\bar{L}(B, \Omega)$, where $B\widehat{\Omega} \in \text{T}_B \text{SO}(3)$. Lagrange–d’Alembert Principle in the invariant form gives rise to the Euler–Lagrange equation,

$$\frac{d}{dt} \frac{\partial \bar{L}}{\partial \Omega} + \Omega \times \frac{\partial \bar{L}}{\partial \Omega} - (B^\top D_B \bar{L})^\vee - \alpha = 0, \quad (10)$$

where α is the exterior force.

To illustrate the above notions, an example is given as follows.

Example 1. Consider a system consisting of a rigid body moving in a central gravitational field. Let E denote the center of a gravitational field with gravitational constant μ , and let O denote the center of mass of the assembly. We attach an inertial reference frame to E and a body reference frame to O , respectively. The radius vector from E to O is represented by $\mathbf{r}$ in the inertial frame. The attitude of the rigid body can be modeled by an element $B \in \text{SO}(3)$, which represents the rotation of the body frame relative to the inertial frame. Let Ω denote the angular velocity of the rigid body expressed in the body frame. With the above notations, the Lagrangian of the system is

$$\bar{L}(B, \Omega) = \frac{1}{2} m |\dot{\mathbf{r}}|^2 + \frac{1}{2} \langle \Omega, \mathbf{I} \Omega \rangle + \int_{\mathcal{B}} \frac{\mu}{|\mathbf{r} + B\mathbf{q}|} dm(\mathbf{q})$$

where $\mathbf{I}$ denotes the inertia tensor of the rigid body. In the term corresponding to the gravitational potential energy, the vector $\mathbf{q}$ (in body coordinates) is from O to a mass element of the rigid body. To apply Eq. (10), one needs to obtain $D_B \bar{L}$. To this end, we have

$$\langle D_B \bar{L}(B, \mathbf{r}), B\hat{a} \rangle = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{\mathcal{B}} \frac{\mu}{|\mathbf{r} + B e^{\varepsilon \hat{a}} \mathbf{q}|} dm(\mathbf{q}) = \int_{\mathcal{B}} \frac{\mu (B^\top \mathbf{r} \times \mathbf{q})^\top \hat{a}}{|\mathbf{r} + B\mathbf{q}|^3} dm(\mathbf{q}),$$

for arbitrary $\hat{a} \in \mathbb{R}^3$. Therefore, it follows that

$$D_B \bar{L}(B, \mathbf{r}) = B \int_{\mathcal{B}} \frac{-\mu (B^\top \mathbf{r} \times \mathbf{q})^\wedge}{|\mathbf{r} + B\mathbf{q}|^3} dm(\mathbf{q}),$$

which is the so-called gravity-gradient torque in integral form. Following Eq. (10), the dynamical equations of the system is

$$\dot{B} = B\widehat{\Omega},$$

$$\mathbf{I} \dot{\Omega} = -\Omega \times \mathbf{I} \Omega + \int_{\mathcal{B}} \frac{\mu (B^\top \mathbf{r} \times \mathbf{q})}{|\mathbf{r} + B\mathbf{q}|^3} dm(\mathbf{q}),$$

$$m \ddot{\mathbf{r}} = - \int_{\mathcal{B}} \frac{\mu (\mathbf{r} + B\mathbf{q})}{|\mathbf{r} + B\mathbf{q}|^3} dm(\mathbf{q}).$$

Recall that the representation of 1-form $D_B \tilde{L}$ is $B\hat{a}$, cf. [6], for some $a \in \mathbb{R}^3$, and one observes that $B^\top D_B \tilde{L}$ is a skew-symmetric matrix so that the third term on the left-hand side of Eq. (10) is well defined. Theorem 2 is the global version of the Euler–Lagrange equation in terms of the rotation matrix B and the angular velocities Ω , whereas local versions can be found in [7, 8, 14]. A similar but weaker result can be found in [9], where some specific forms of $\tilde{L}$ were imposed. (For instance, the gravitational potential energy function mentioned above does not satisfy the requirement in [9].) Moreover, it is noteworthy that the derivation of Theorem 2 here is carried out under the framework of global representation of $SO(3)$ and the invariant formulation of Lagrangian mechanics. Such formula is applicable to a large class of physical system including aircraft, spacecraft, space robot, rigid body, and multi-body systems.

4. The skew-symmetric property

Recall that a Lagrangian system is called *natural* or *simple* if the associated Lagrangian equals to the difference between the kinetic and the potential energies [1, 2]. Here, we consider a larger class of mechanical systems represented by the Lagrangian:

$$L(x, v) = \frac{1}{2} v^\top M(x) v + Y(x)^\top v - V(x), \quad (11)$$

where the Lagrangian L is in terms of generalized coordinates $x = (x_1, \dots, x_n)$ and generalized velocities $v = (v_1, \dots, v_n)$. In the above equation, $M(x)$ and $V(x)$ denote the mass inertia tensor and a potential, respectively. The second term including the gyroscopic vector $Y(x)$ is linear in the velocity variable v and is responsible for the paradoxical behavior of the gyroscopes [13]. Accordingly, natural Lagrangian systems, i.e., with the gyroscopic term absent, are a sub-class of the mechanical systems considered here. To obtain the dynamical equations associated with the Lagrangian in Eq. (11), the Euler–Lagrange equations (4) can be invoked. First, we find $\partial L / \partial v = M(x)v + Y(x)$, and then apply the chain rule of a matrix derivative [12] and the identity $v = \dot{x}$ to get

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial v} &= M(x)\dot{v} + \left(\frac{\partial M(x)}{\partial x^\top} (v \otimes \mathbf{1}) \right) v + \frac{\partial Y(x)}{\partial x^\top} v \\ &= M(x)\dot{v} + \frac{1}{2} \left(\frac{\partial M(x)}{\partial x^\top} (v \otimes \mathbf{1}) + \frac{\partial M(x)}{\partial x^\top} (\mathbf{1} \otimes v) \right) v + \frac{\partial Y(x)}{\partial x^\top} v, \end{aligned} \quad (12)$$

where the property $(v \otimes \mathbf{1})v = (\mathbf{1} \otimes v)v$ has been used. Here, the notation $\otimes$ denotes the Kronecker product and $\mathbf{1}$ denotes the identity matrix with suitable dimension. Next, the partial derivative of L with respect to x is obtained by using the product rule [12] as

$$\frac{\partial L}{\partial x} = \frac{1}{2} (\mathbf{1} \otimes v^\top) \frac{\partial M(x)^\top}{\partial x} v + \frac{\partial Y(x)^\top}{\partial x} v - \frac{\partial V(x)}{\partial x}. \quad (13)$$

Substituting Eqs. (12) and (13) into the Euler–Lagrange equations (4), we obtain the following theorem.

Theorem 3. *The equations of motion for a mechanical system with Lagrangian of the form (11) can be expressed as*

$$M(x)\ddot{x} + (\mathcal{C} + \mathcal{G})\dot{x} + \frac{\partial V(x)}{\partial x} = \alpha, \quad (14)$$

with

$$\begin{aligned} \mathcal{C} &\triangleq \frac{1}{2} \left(\frac{\partial M(x)}{\partial x^\top} (v \otimes \mathbf{1}) + \frac{\partial M(x)}{\partial x^\top} (\mathbf{1} \otimes v) - (\mathbf{1} \otimes v^\top) \frac{\partial M(x)^\top}{\partial x} \right), \\ \mathcal{G} &\triangleq \frac{\partial Y(x)}{\partial x^\top} - \frac{\partial Y(x)^\top}{\partial x}. \end{aligned}$$

Moreover, $\dot{M}(x) - 2(\mathcal{C} + \mathcal{G})$ is a skew-symmetric matrix.

Proof. The matrix $\mathcal{G}$ is obviously skew-symmetric. The time derivative of $M(x)$ appeared in Eq. (12) makes

$$\dot{M}(x) - 2\mathcal{G} = - \left(\frac{\partial M(x)}{\partial x^\top} (\mathbf{1} \otimes v) - (\mathbf{1} \otimes v^\top) \frac{\partial M(x)^\top}{\partial x} \right).$$

It then follows that $\dot{M} - 2\mathcal{G}$ is skew-symmetric. $\square$

The skew-symmetric property for systems on $\mathbb{R}^n$ is useful in designing adaptive controllers. It is worthy to note that the dynamic equations of a robot manipulator (cf., in [10]) are similar to Eq. (14), but without the gyroscopic term $\mathcal{G}$. Compared to the complicated indicial notations adopted in [10], the concise symbolic forms for $\mathcal{C}$ and $\mathcal{G}$ are explicitly given here to reveal this property.

We now focus on the skew-symmetric property of mechanical systems with configuration space $\text{SO}(3)$. Let $(B, B\hat{\Omega})$ denote an element in $\text{T}\text{SO}(3)$. The Lagrangian of the corresponding mechanical system can be expressed as, cf. Eq. (11),

$$L(B, B\hat{\Omega}) = \frac{1}{2} \Omega^\top M(B) \Omega + \Omega^\top Y(B) - V(B) \triangleq \bar{L}(B, \Omega). \quad (15)$$

Substituting $\partial \bar{L} / \partial \Omega = M(B) \Omega + Y(B)$ into Eq. (10) yields

$$M \dot{\Omega} + (\dot{M} - \widehat{M\Omega} - \hat{Y}) \Omega + \dot{Y} - (\frac{1}{2} B^\top D_B (\Omega^\top M \Omega) + B^\top D_B (\Omega^\top Y) - B^\top D_B V)^\vee = \alpha,$$

which involves the terms $D_B(\Omega^\top M \Omega)$ and $D_B(\Omega^\top Y)$. By definition,

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \Omega^\top M(B e^{\varepsilon \hat{\Omega}}) \Omega = \langle D_B(\Omega^\top M \Omega), B\hat{\Omega} \rangle,$$

where the right-hand side equals to $\langle B\hat{w}_1, B\hat{\Omega} \rangle = \Omega^\top w_1$ for some unknown $w_1 \in \mathbb{R}^3$, which implies

$$w_1 = (B^\top D_B(\Omega^\top M \Omega))^\vee = \dot{M} \Omega + \Omega_a^\perp, \quad (16)$$

with some vector $\Omega_a^\perp$ which is orthogonal (with respect to Euclidean inner product) to Ω . Note that the property

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \Omega^\top M(B e^{\varepsilon \hat{\Omega}}) \Omega = \Omega^\top \dot{M} \Omega,$$

(where $\dot{B} = B\hat{\Omega}$) has been used in the derivation. Similarly, we have

$$\Omega^\top \dot{Y} = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \Omega^\top Y(B e^{\varepsilon \hat{\Omega}}) = \langle D_B(\Omega^\top Y), B\hat{\Omega} \rangle,$$

and hence

$$(B^\top D_B(\Omega^\top Y))^\vee = \dot{Y} + \Omega_b^\perp, \quad (17)$$

with some vector $\Omega_b^\perp$ orthogonal to Ω . The following lemma states that $\Omega_a^\perp$ and $\Omega_b^\perp$ can be expressed as the products of a skew-symmetric matrix and Ω .

Lemma 1. In $\mathbb{R}^n$, every vector orthogonal to vector u can always be expressed as Au , for some skew-symmetric matrix $A \in \mathbb{R}^{n \times n}$.

Proof. Let such a vector be denoted as $u^\perp$. Without loss of generality, let u be a unit vector. For $n=1$, A is trivially chosen as 0. For $n \geq 2$, there exists an orthogonal transformation T through Gram–Schmidt

orthogonalization process, such that Tu can be expressed as $(1, 0, \dots, 0)^\top$ with $Tu^\perp$ being $(0, a_2, a_3, \dots, a_n)^\top$ for some $a_2, a_3, \dots, a_n$. Then, one choose $\bar{A}$ as

$$\bar{A} = \begin{pmatrix} 0 & -a_2 & \cdots & -a_n \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix},$$

so that $\bar{A}Tu = Tu^\perp$. Hence, $A = T^\top \bar{A} T$ is a desired skew-symmetric matrix satisfying $u^\perp = Au$. $\square$

By the above lemma and Eqs. (16) and (17), there exist $a_1, a_2 \in \mathbb{R}^3$ such that $w_1 = \dot{M}\Omega + \hat{a}_1\Omega$ and $(B^\top D_B(\Omega^\top Y))^\vee = \dot{Y} + \hat{a}_2\Omega$. As a result, the equations of motion can be expressed as follows.

Theorem 4. *The equations of motion for a mechanical system with Lagrangian of the form (15) can be expressed as*

$$M\dot{\Omega} + (\mathcal{C} + \mathcal{G})\Omega + (B^\top D_B V)^\vee = \alpha, \quad (18)$$

with

$$\mathcal{C} = \frac{1}{2}\dot{M} - \widehat{M\Omega} - \frac{1}{2}\hat{a}_1 \quad \text{and} \quad \mathcal{G} = -\hat{Y} - \hat{a}_2,$$

for some a_1, a_2 , and, furthermore, $\dot{M} - 2(\mathcal{C} + \mathcal{G})$ is a skew-symmetric matrix.

The skew-symmetric property is easily observed from the expression of $\mathcal{C}$ and $\mathcal{G}$. Hence, the equations of motion for the mechanical systems in the form of Eq. (18) also exhibit the skew-symmetric property. Next, we generalize the above formula to the case with motion in $\text{SO}(3)^m \times \mathbb{R}^n$, whereby the configuration space for many coupled multi-rigid-body systems can be modeled. Let $B = \text{diag}\{B_1, B_2, \dots, B_m\}$ and $\Omega = (\Omega_1^\top, \Omega_2^\top, \dots, \Omega_m^\top)^\top$, which are $(3m \times 3m)$ matrix and $(3m \times 1)$ vector, respectively, with $\dot{B}_i = B_i \hat{\Omega}_i$, $i = 1, \dots, m$. Define $\hat{\Omega}$ as $\text{diag}\{\hat{\Omega}_1, \hat{\Omega}_2, \dots, \hat{\Omega}_m\}$, a $(3m \times 3m)$ matrix. Accordingly, we have the kinematic equation on $\text{SO}(3)^m$, that is, $\dot{B} = B\hat{\Omega}$. Let x be a vector in $\mathbb{R}^n$. The velocity phase variables of $\text{SO}(3)^m \times \mathbb{R}^n$ can be defined as (Q, v) , that is,

$$Q \triangleq (B, x) \quad \text{and} \quad v \triangleq \begin{pmatrix} \Omega \\ v \end{pmatrix},$$

with $\dot{x} = v$. Note that Q simply denotes an element in the configuration space, while v is indeed a vector in $\mathbb{R}^{3m+n}$. Analogous to Eqs. (11) and (15), the general mechanical system on $\text{SO}(3)^m \times \mathbb{R}^n$ is defined to be a mechanical system with Lagrangian:

$$\bar{L}(Q, v) = \frac{1}{2}v^\top Mv + v^\top Y(Q) - V(Q). \quad (19)$$

One can easily extend the equation in Theorem 2 to the case in which the configuration space is $(\text{SO}(3))^m \times \mathbb{R}^n$ so that the following Euler–Lagrange equation can be obtained:

$$\frac{d}{dt} \frac{\partial \bar{L}}{\partial v} - \left(\frac{(\partial \bar{L} / \partial \Omega)^\wedge \Omega + (B^\top D_B \bar{L})^\vee}{\partial \bar{L} / \partial x} \right) - \alpha = 0, \quad (20)$$

where $D_B \bar{L}$ is understood as $\text{diag}\{D_{B_1} \bar{L}, \dots, D_{B_m} \bar{L}\}$, and $(\partial \bar{L} / \partial \Omega)^\wedge = \text{diag}\{(\partial \bar{L} / \partial \Omega_1)^\wedge, \dots, (\partial \bar{L} / \partial \Omega_m)^\wedge\}$, which is a $(3m \times 3m)$ matrix. Applying similar techniques for the case of $\text{SO}(3)$, the skew-symmetric property in parallel to Theorem 4 for the general system can be derived. From

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} v^\top M(Be^{\varepsilon \hat{\Omega}}, x + \varepsilon v)v = \langle D_B(v^\top Mv), B\hat{\Omega} \rangle + \left\langle \frac{\partial v^\top Mv}{\partial x}, v \right\rangle,$$

the right-hand side can be realized as

$$\langle \widehat{B\mathbf{w}_1}, \widehat{B\hat{\Omega}} \rangle + \langle \mathbf{w}_2, \mathbf{v} \rangle = \mathbf{v}^\top \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix},$$

for some unknown vectors $\mathbf{w}_1 \in \mathbb{R}^{3m}$ and $\mathbf{w}_2 \in \mathbb{R}^n$. It can be checked that

$$\begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{B}^\top \mathbf{D}_B(\mathbf{v}^\top M \mathbf{v}))^\vee \\ \partial \mathbf{v}^\top M \mathbf{v} / \partial \mathbf{x} \end{pmatrix} = \dot{M} \mathbf{v} + \mathbf{v}_a^\perp,$$

with some vector $\mathbf{v}_a^\perp$ orthogonal to $\mathbf{v}$. Similarly, one has

$$\begin{pmatrix} (\mathbf{B}^\top \mathbf{D}_B(\mathbf{v}^\top Y))^\vee \\ \partial \mathbf{v}^\top Y / \partial \mathbf{x} \end{pmatrix} = \dot{Y} + \mathbf{v}_b^\perp,$$

for some vector $\mathbf{v}_b^\perp$ orthogonal to $\mathbf{v}$. Accordingly, the equations of motion for the Lagrangian (19) are expressed as

$$M\dot{\mathbf{v}} + \frac{1}{2}\dot{M}\mathbf{v} - \begin{pmatrix} \widehat{\frac{\partial \bar{L}}{\partial \hat{\Omega}}} \hat{\Omega} \\ 0 \end{pmatrix} + \mathbf{v}_c^\perp + \mathbf{D}_Q V - \alpha = 0,$$

where $\mathbf{v}_c^\perp = \frac{1}{2}\mathbf{v}_a^\perp + \mathbf{v}_b^\perp$ is also orthogonal to $\mathbf{v}$. With the aid of Lemma 1, the following theorem is obtained.

Theorem 5. *The equations of motion for a mechanical system with Lagrangian of the form (19) can be expressed as*

$$M(Q)\dot{\mathbf{v}} + \mathcal{D}(Q, \mathbf{v})\mathbf{v} + \begin{pmatrix} (\mathbf{B}^\top \mathbf{D}_B V(Q))^\vee \\ \mathbf{D}_x V(Q) \end{pmatrix} = \alpha, \quad (21)$$

in which the matrix formed by $\dot{M}(Q) - 2\mathcal{D}(Q, \mathbf{v})$ is a skew-symmetric matrix.

There are many physical systems belonging to the class of gyroscopic systems. One example is a spacecraft system with control moment gyros (CMGs), whose Lagrangian can be described by

$$\bar{L}(Q, \mathbf{v}) = \frac{1}{2}\mathbf{v}^\top M \mathbf{v} + \mathbf{v}^\top Y + \int_{\mathcal{B}} \frac{\mu}{|\mathbf{r} + B\mathbf{q}|} dm(\mathbf{q}),$$

where the second term is the gyroscopic term arising from the constant speed rotation of CMGs and the third term represents the potential energy. Here, $\mathbf{v}$ denotes the stack of the angular velocity of the spacecraft, the angular velocities of gimbals, and the spacecraft (linear) velocity. For other examples, cf. [13].

Example 2. As an example to demonstrate the dynamics derivation and the skew-symmetric property, here we consider a system consisting of two rigid bodies connected by a ball-in-socket joint. A sketch of the system is found in Fig. 1. Only the attitude dynamics is considered and the configuration space of the system is thus $\text{SO}(3)^2$. Let $\{b_1, b_2, b_3\}$ be the body frame located at the center of the rigid body and $\{c_1, c_2, c_3\}$ be the link frame located at the base of the link. The vectors from the center of mass of the link to its base and from $\{b_1, b_2, b_3\}$ to $\{c_1, c_2, c_3\}$ are denoted by $\mathbf{l}$ and $\mathbf{d}$, respectively. Let $\sigma = (m_0 m_1)/(m_0 + m_1)$, where m_0 and m_1 are the masses of the rigid bodies, respectively. The Lagrangian of the system is

$$\begin{aligned} L(R, \Omega, \omega) &= \frac{1}{2}\mathbf{v}^\top \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \mathbf{v} \\ &= \frac{1}{2} \begin{pmatrix} \Omega \\ \omega \end{pmatrix}^\top \begin{pmatrix} \mathbf{I} - \sigma \hat{\mathbf{d}} \hat{\mathbf{d}} - 2\sigma \hat{\mathbf{d}} \hat{\mathbf{R}} \mathbf{l} + \mathbf{I} - \sigma \hat{\mathbf{l}} \hat{\mathbf{l}} & -\sigma \hat{\mathbf{d}} \hat{\mathbf{R}} \mathbf{l} + \mathbf{R}(\mathbf{I} - \sigma \hat{\mathbf{l}} \hat{\mathbf{l}}) \\ -\sigma \hat{\mathbf{l}} \mathbf{R}^\top \hat{\mathbf{d}} + (\mathbf{I} - \sigma \hat{\mathbf{l}} \hat{\mathbf{l}}) \mathbf{R}^\top & \mathbf{I} - \sigma \hat{\mathbf{l}} \hat{\mathbf{l}} \end{pmatrix} \begin{pmatrix} \Omega \\ \omega \end{pmatrix}, \end{aligned}$$

Fig. 1. Two rigid bodies connected by a ball-in-socket joint.

where I and I , B and R , and Ω and ω are the moments of inertia tensor, the attitudes, and angular velocities of the rigid body and the link, respectively. It is noted that B is cyclic in the Lagrangian. To derive the dynamics, one needs to compute $(B^T D_B L)^\vee$, which can be shown as

$$\begin{pmatrix} (B^T D_B L)^\vee \\ (R^T D_R L)^\vee \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \sigma \hat{l} R^T \hat{\Omega} \hat{d} + U & 0 \end{pmatrix} \begin{pmatrix} \Omega \\ \omega \end{pmatrix},$$

where $U = -\sigma \hat{l} (R\hat{\omega})^T \hat{d} + (I - \sigma \hat{l} \hat{l})(R\hat{\omega})^T$. Substituting this equation and the following one:

$$\dot{M}v = \begin{pmatrix} -2\sigma \hat{d} R \hat{\omega} \hat{l} & U^T \\ U & 0 \end{pmatrix} v = \begin{pmatrix} -\sigma \hat{d} R \hat{\omega} \hat{l} & -\sigma \hat{d} \hat{\Omega} R \hat{l} + U^T \\ U & 0 \end{pmatrix} v$$

into Eq. (20), the dynamical equations are derived:

$$M\dot{v} + \begin{pmatrix} -\sigma \hat{d} R \hat{\omega} \hat{l} - (M_{11}\Omega + M_{12}\omega)^\wedge & -\sigma \hat{d} \hat{\Omega} R \hat{l} + U^T \\ -\sigma \hat{l} R^T \hat{\Omega} \hat{d} & -(M_{21}\Omega + M_{22}\omega)^\wedge \end{pmatrix} v = \alpha.$$

Comparing it with Eq. (21), the skew-symmetric property can be justified by the following skew symmetric matrix:

$$\dot{M} - 2\mathcal{D} = \begin{pmatrix} 2(M_{11}\Omega + M_{12}\omega)^\wedge & 2\sigma \hat{d} \hat{\Omega} R \hat{l} - U^T \\ U + 2\sigma \hat{l} R^T \hat{\Omega} \hat{d} & 2(M_{21}\Omega + M_{22}\omega)^\wedge \end{pmatrix}.$$

5. Conclusions

The globally valid formula of Euler–Lagrange equation on the tangent space of $SO(3)$ was extended to obtain the dynamics of mechanical systems with configuration space $SO(3)^m \times \mathbb{R}^n$. A skew-symmetric property was proved rigorously for these dynamical equations. This property has many potential applications in engineering problems, especially in the controller design of robots and spacecraft systems.

References

- [1] R. Abraham, J.E. Marsden, *Foundations of Mechanics*, 2nd ed., Benjamin/Cummings, Menlo Park, CA, 1978.
- [2] V.I. Arnold, *Mathematical Method of Classical Mechanics*, 2nd ed., Springer, Berlin, 1988.
- [3] O. Egeland, J.R. Sagli, Coordination of motion in a spacecraft/manipulator system, *Int. J. Robotics Res.* 12 (1993) 366–379.
- [4] O. Egeland, J.-M. Godhavn, Passivity-based adaptive attitude control of a rigid spacecraft, *IEEE Trans. Autom. Control* 39 (1994) 842–846.
- [5] K.-Y. Lian, L.-S. Wang, L.-C. Fu, Controllability of spacecraft system in a central gravitational field, *IEEE Trans. Autom. Control* 39 (1994) 2426–2441.
- [6] K.-Y. Lian, L.-S. Wang, L.-C. Fu, Globally-valid adaptive controllers of mechanical systems, *IEEE Trans. Autom. Control* 42 (1997) 1149–1154.
- [7] P.W. Linkins, Quasicoordinate equations for flexible structure, *AIAA J.* 13 (1975) 524–526.
- [8] L. Meirovitch, *Computational Methods in Structure Dynamics*, Sijthoff and Noordhoff, Netherlands, 1980.
- [9] R.D. Quinn, Equations of motion for structures in terms of quasi-coordinated, *ASME J. Appl. Mech.* 57 (1990) 745–749.
- [10] J. Slotine, W. Li, On the adaptive control of robot manipulators, *Int. J. Robotics Res.* 6 (1986) 49–59.
- [11] A.M. Vershik, L.D. Faddeev, Lagrangian mechanics in invariant form, *Sel. Math. Sov.* 4 (1981) 339–350.
- [12] W.J. Vetter, Derivative operations on matrices, *IEEE Trans. Autom. Control* 15, 16 (1970) 57–65, 113.
- [13] L.-S. Wang, P.S. Krishnaprasad, Gyroscopic control and stabilization, *J. Nonlinear Sci.* 2 (1992) 367–415.
- [14] E.T. Whittaker, *Analytical Dynamics of Particles and Rigid Bodies*, Dover, New York, 1944.
- [15] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles: Differential Geometry*, Marcel Dekker, New York, 1973.