Stability of discrete-time uncertain systems with a time-varying state delay

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Abstract: The stability analysis problem for unforced discrete-time systems with a time-varying state delay is considered. Stability criteria for nominal and uncertain system models are derived such that the upper and lower bounds on the delay time can be determined to ensure asymptotic stability. There is no need to assume that the system is stable when the delay vanishes. Adopting the linear matrix inequality approach enables the computations to be implemented conveniently. Compared with existing results in the literature, the proposed method is less conservative for many cases.

Keywords: linear matrix inequality, discrete-time system, time-varying delay, stability

1 INTRODUCTION

Time delay occurs in electrical, mechanical, economic, and biological systems, and is a well-known cause of instability and performance degradation. In recent years, increasing attention has been paid to the problems of stability analysis and controller design for time delay systems. This is particularly necessary in the field of chemical process control, where time delay is almost inevitable, and controllers are sought to enhance system stability and performance [1–4].

After controllers are designed for nominal systems, one often has to check the robustness with respect to all parameter uncertainties in the system model. If the time delay is also uncertain, with its true value being different from its nominal value, then the system's robust stability should be confirmed for a range of time delay values [5]. The stability problem has been investigated in the literature for systems with an interval time delay [6, 7], and in certain cases it has been found that some controllers can tolerate less time delay variation than others [8, 9].

Although there are already many stability analysis methods proposed for time delay systems, including delay-independent methods [10, 11] and delay-dependent methods [12–23], most start with the assumption that the system is stable when the time delay is zero, and try to find out the full extent of the tolerance to the time delay. The assumption of stability with no time delay is even adopted implicitly in some results that discuss the stability with respect to intervals of time delay that exclude zero [21–23]. Only a few studies have assumed that the system to be analysed is known to be stable for a non-zero time delay [5]. However, as stated above, after a controller is designed to stabilize a system, the closed-loop system is usually known to be stable for the specific nominal time delay. Furthermore, the controller may not be robust enough to cover such a large time delay range as to ensure that the closed-loop system is stable when the time delay is shortened to zero. Thus, assuming stability at a zero-valued time delay may not be practical.

In this paper, the stability analysis problem for uncertain and nominal unforced discrete-time systems with a time-varying state delay is studied. Stability conditions are derived based on a Lyapunov function defined for an augmented system. It is assumed that the system is asymptotically stable with a specific time delay, and upper bounds on the

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allowable additional time delay are derived. The stability criteria are formulated using the linear matrix inequality (LMI) approach [24], which is convenient to check. Numerical examples are provided to illustrate the application of the developed results, and comparisons with results from existing methods are made to show that the proposed method is not only less conservative for the considered examples, but also applicable to systems which are not stable when the time delay is zero.

2 PROBLEM FORMULATION

Consider the following uncertain discrete-time delay system

$$\begin{aligned} \mathbf{x}(k+1) &= (\mathbf{A} + \Delta\mathbf{A}(k))\mathbf{x}(k) + (\mathbf{A}_d + \Delta\mathbf{A}_d(k))\mathbf{x}(k-h(k)) \\ \mathbf{x}(k) &= \phi(k), \quad \text{for } -h_2 \leq k \leq 0 \end{aligned} \quad (1)$$

where k is the discrete-time index, $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{A}$ and $\mathbf{A}_d$ are real $n \times n$ parameter matrices, and $h(k)$ is a non-negative time delay depending on k . It is assumed that

$$0 \leq h_1 \leq h(k) \leq h_2 \quad (2)$$

where h_1 and h_2 are positive integers, and the initial conditions of the system are specified by the given vector function $\phi(\cdot)$ in equation (1). The two real $n \times n$ matrix functions $\Delta\mathbf{A}(\cdot)$ and $\Delta\mathbf{A}_d(\cdot)$ represent time-varying parameter uncertainties. It is further assumed that the uncertainties are norm-bounded and can be described as

$$\Delta\mathbf{A}(k) = \mathbf{D}\mathbf{F}(k)\mathbf{E}_a, \quad \Delta\mathbf{A}_d(k) = \mathbf{D}\mathbf{F}(k)\mathbf{E}_d \quad (3)$$

where $\mathbf{D}$, $\mathbf{E}_a$, and $\mathbf{E}_d$ are real constant matrices with appropriate dimensions and $\mathbf{F}(k)$ is an unknown matrix satisfying

$$\mathbf{F}^T(k)\mathbf{F}(k) \leq \mathbf{I} \quad (4)$$

When the uncertainties in the system matrices are ignored, a nominal system of equation (1) is

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{A}_d\mathbf{x}(k-h(k)) \\ \mathbf{x}(k) &= \phi(k), \quad \text{for } -h_2 \leq k \leq 0 \end{aligned} \quad (5)$$

To find the delay-dependent stability criteria for discrete-time delay systems such as (5), it is often assumed that the system is asymptotically stable when

$h(k) = 0$. This is explicitly expressed in [25], and also in [26] and [27] in order to discuss the continuous-time case, but it is often not mentioned in other related works, although the assumption is actually necessary therein. Here for a broader investigation purpose, the system (5) is assumed to be asymptotically stable when $h(k) = h_1$. In order to handle this kind of situation, an augmented state vector

$$\mathbf{z}(k) = \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{x}(k-1) \\ \vdots \\ \mathbf{x}(k-h_1) \end{bmatrix} \quad (6)$$

and its delayed term

$$\mathbf{z}(k+h_1-h(k)) = \begin{bmatrix} \mathbf{x}(k+h_1-h(k)) \\ \mathbf{x}(k-1+h_1-h(k)) \\ \vdots \\ \mathbf{x}(k-h(k)) \end{bmatrix}$$

are defined, and the augmented system

$$\mathbf{z}(k+1) = \tilde{\mathbf{A}}\mathbf{z}(k) + \tilde{\mathbf{A}}_d\mathbf{z}(k+h_1-h(k)) \quad (7)$$

is discussed, where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{A}}_d = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_d \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

are matrices of compatible dimensions. The problem here is to find h_2 to guarantee that equation (5) remains asymptotically stable.

In the subsequent development, the following lemma will play an important role, and is presented here. Note in matrix inequalities, the notation “ $>$ ” ($\geq$) “ $<$ ” means positive (semi-)definiteness, “ < 0 ” means negative definiteness, and “ $\mathbf{A} \leq \mathbf{B}$ ” means $\mathbf{A} - \mathbf{B} \leq 0$.

Lemma 1 [17]. Given a symmetric matrix $\mathbf{M}$ and matrices $\mathbf{H}$, $\mathbf{E}$ with appropriate dimensions, then

$$\mathbf{M} + \mathbf{H}\mathbf{F}\mathbf{E} + \mathbf{E}^T\mathbf{F}^T\mathbf{H}^T < 0$$

for all $\mathbf{F}$ satisfying $\mathbf{F}^T\mathbf{F} \leq \mathbf{I}$, if and only if there exists an $\varepsilon > 0$ such that

$$\mathbf{M} + \varepsilon\mathbf{H}\mathbf{H}^T + \varepsilon^{-1}\mathbf{E}^T\mathbf{E} < 0$$

3 STABILITY ANALYSIS

In this section, the Lyapunov stability method will be adopted to find an h_2 for a given h_1 to ensure the stability of the augmented system (7), and thus the stability of system (5). The main result is presented below. For the sake of simplicity, $*$ is used as an ellipsis for the terms that are clear by the symmetry in a block matrix, and a special font such as “ $\mathbb{R}$ ” is used to represent matrix variables.

Theorem 1. The system (7) is asymptotically stable for $0 \leq h_1 \leq h(k) \leq h_2$, if there exist matrices $\mathbb{P} > 0, \mathbb{Q} > 0, \mathbb{R} > 0, \mathbb{Y}$, and $\mathbb{Z}$, all with the size $(h_1 + 1)n \times (h_1 + 1)n$, such that

$$\begin{bmatrix} \phi_{11} + \tilde{\mathbf{A}}^T \mathbb{P} \tilde{\mathbf{A}} + \tau \mathbb{Q} + \tau \hat{\mathbf{A}}^T \mathbb{R} \hat{\mathbf{A}} & \phi_{12} + \tilde{\mathbf{A}}^T \mathbb{P} \tilde{\mathbf{A}}_d + \tau \hat{\mathbf{A}}^T \mathbb{R} \tilde{\mathbf{A}}_d & \tau \mathbb{Y} \\ * & \phi_{22} + \tilde{\mathbf{A}}_d^T \mathbb{P} \tilde{\mathbf{A}}_d + \tau \hat{\mathbf{A}}_d^T \mathbb{R} \tilde{\mathbf{A}}_d & \tau \mathbb{Z} \\ * & * & -\tau \mathbb{R} \end{bmatrix} < 0 \quad (8)$$

where $\hat{\mathbf{A}} = \tilde{\mathbf{A}} - \mathbf{I}$, $\tau = h_2 - h_1$, $\phi_{11} = -\mathbb{P} + \mathbb{Q} + \mathbb{Y} + \mathbb{Y}^T$,

$\phi_{12} = -\mathbb{Y} + \mathbb{Z}^T$, and $\phi_{22} = -\mathbb{Q} - \mathbb{Z} - \mathbb{Z}^T$

Proof. By assumption, equations (5) and (7) are asymptotically stable when $h(k) = h_1$ for all k , therefore only the case with $h_2 > h_1$ needs to be discussed. Let $\mathbf{y}(k) = \mathbf{z}(k+1) - \mathbf{z}(k)$. Then

$$\mathbf{y}(k) = (\tilde{\mathbf{A}} - \mathbf{I})\mathbf{z}(k) + \tilde{\mathbf{A}}_d \mathbf{z}(k + h_1 - h(k)) \quad (9)$$

and

$$\mathbf{z}(k + h_1 - h(k)) = \mathbf{z}(k) - \sum_{i=k+h_1-h(k)}^{k-1} \mathbf{y}(i) \quad (10)$$

Choose a Lyapunov function candidate as

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) \quad (11)$$

where

$$V_1(k) = \mathbf{z}^T(k) \mathbb{P} \mathbf{z}(k), \quad V_2(k) = \sum_{i=k+h_1-h(k)}^{k-1} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i),$$

$$V_3(k) = \sum_{i=1+h_1-h_2}^0 \sum_{m=k+i}^{k-1} \mathbf{z}^T(m) \mathbb{Q} \mathbf{z}(m),$$

$$V_4(k) = \sum_{i=h_1-h_2}^{-1} \sum_{m=k+i}^{k-1} \mathbf{y}^T(m) \mathbb{R} \mathbf{y}(m)$$

and $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ are positive definite matrices to be determined. Consider the Lyapunov difference

$\Delta V_i(k) = V_i(k+1) - V_i(k)$ for $i = 1, 2, 3, 4$. Firstly,

$$\begin{aligned} \Delta V_1(k) &= \mathbf{z}^T(k+1) \mathbb{P} \mathbf{z}(k+1) - \mathbf{z}^T(k) \mathbb{P} \mathbf{z}(k) \\ &= \mathbf{z}^T(k) (\tilde{\mathbf{A}}^T \mathbb{P} \tilde{\mathbf{A}} - \mathbb{P}) \mathbf{z}(k) \\ &\quad + 2\mathbf{z}^T(k) (\tilde{\mathbf{A}}^T \mathbb{P} \tilde{\mathbf{A}}_d) \mathbf{z}(k + h_1 - h(k)) \\ &\quad + \mathbf{z}^T(k + h_1 - h(k)) (\tilde{\mathbf{A}}_d^T \mathbb{P} \tilde{\mathbf{A}}_d) \mathbf{z}(k + h_1 - h(k)) \end{aligned} \quad (12)$$

Secondly

$$\begin{aligned} \Delta V_2(k) &= \sum_{i=k+1+h_1-h(k+1)}^k \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) \\ &\quad - \sum_{i=k+h_1-h(k)}^{k-1} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) \\ &= \mathbf{z}^T(k) \mathbb{Q} \mathbf{z}(k) \\ &\quad - \mathbf{z}^T(k + h_1 - h(k)) \mathbb{Q} \mathbf{z}(k + h_1 - h(k)) \\ &\quad + \sum_{i=k+1+h_1-h(k+1)}^{k-1} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) \\ &\quad - \sum_{i=k+1+h_1-h(k)}^{k-1} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) \end{aligned} \quad (13)$$

where

$$\begin{aligned} \sum_{i=k+1+h_1-h(k+1)}^{k-1} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) &\leq \sum_{i=k+1+h_1-h_2}^{k-1} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) \\ &= \sum_{i=k+1+h_1-h_2}^{k+h_1-h(k)} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) + \sum_{i=k+1+h_1-h(k)}^{k-1} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) \end{aligned} \quad (14)$$

Hence

$$\begin{aligned} \Delta V_2(k) &\leq \sum_{i=k+1+h_1-h_2}^{k+h_1-h(k)} \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) + \mathbf{z}^T(k) \mathbb{Q} \mathbf{z}(k) \\ &\quad - \mathbf{z}^T(k + h_1 - h(k)) \mathbb{Q} \mathbf{z}(k + h_1 - h(k)) \end{aligned} \quad (15)$$

Thirdly

$$\begin{aligned} \Delta V_3(k) &= \sum_{i=1+h_1-h_2}^0 \left(\sum_{m=k+1+i}^k \mathbf{z}^T(m) \mathbb{Q} \mathbf{z}(m) \right. \\ &\quad \left. - \sum_{m=k+i}^{k-1} \mathbf{z}^T(m) \mathbb{Q} \mathbf{z}(m) \right) \\ &= \tau \mathbf{z}^T(k) \mathbb{Q} \mathbf{z}(k) - \sum_{i=k+1+h_1-h_2}^k \mathbf{z}^T(i) \mathbb{Q} \mathbf{z}(i) \end{aligned} \quad (16)$$

where

$$\begin{aligned} \sum_{i=k+1+h_1-h_2}^k \mathbf{z}^T(i) \mathbf{Q} \mathbf{z}(i) &= \\ \sum_{i=k+1+h_1-h_2}^{k+h_1-h_1} \mathbf{z}^T(i) \mathbf{Q} \mathbf{z}(i) &\geq \sum_{i=k+1+h_1-h_2}^{k+h_1-h(k)} \mathbf{z}^T(i) \mathbf{Q} \mathbf{z}(i) \end{aligned} \quad (17)$$

Thus

$$\Delta V_3(k) \leq \tau \mathbf{z}^T(k) \mathbf{Q} \mathbf{z}(k) - \sum_{i=k+1+h_1-h_2}^{k+h_1-h(k)} \mathbf{z}^T(i) \mathbf{Q} \mathbf{z}(i) \quad (18)$$

Finally

$$\begin{aligned} \Delta V_4(k) &= \sum_{i=h_1-h_2}^{-1} \left(\sum_{m=k+1+i}^k \mathbf{y}^T(m) \mathbf{R} \mathbf{y}(m) \right. \\ &\quad \left. - \sum_{m=k+i}^{k-1} \mathbf{y}^T(m) \mathbf{R} \mathbf{y}(m) \right) \\ &= \tau \mathbf{y}^T(k) \mathbf{R} \mathbf{y}(k) - \sum_{i=k+h_1-h_2}^{k-1} \mathbf{y}^T(i) \mathbf{R} \mathbf{y}(i) \end{aligned} \quad (19)$$

where

$$\sum_{i=k+h_1-h(k)}^{k-1} \mathbf{y}^T(i) \mathbf{R} \mathbf{y}(i) \leq \sum_{i=k+h_1-h_2}^{k-1} \mathbf{y}^T(i) \mathbf{R} \mathbf{y}(i) \quad (20)$$

Thus, with the relation in equation (9)

$$\begin{aligned} \Delta V_4(k) &\leq \mathbf{z}^T(k) \left(\tau \hat{\mathbf{A}}^T \mathbf{R} \hat{\mathbf{A}} \right) \mathbf{z}(k) \\ &\quad + 2 \mathbf{z}^T(k) \left(\tau \hat{\mathbf{A}}^T \mathbf{R} \tilde{\mathbf{A}}_d \right) \mathbf{z}(k+h_1-h(k)) \\ &\quad + \mathbf{z}^T(k+h_1-h(k)) \left(\tau \tilde{\mathbf{A}}_d^T \mathbf{R} \tilde{\mathbf{A}}_d \right) \mathbf{z}(k+h_1-h(k)) \\ &\quad - \sum_{i=k+h_1-h(k)}^{k-1} \mathbf{y}^T(i) \mathbf{R} \mathbf{y}(i) \end{aligned} \quad (21)$$

From equations (12), (15), (18), and (21)

$$\begin{aligned} \Delta V(k) &= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k) \\ &\leq \xi^T(k) \mathbf{\Omega} \xi(k) - \sum_{i=k+h_1-h(k)}^{k-1} \mathbf{y}^T(i) \mathbf{R} \mathbf{y}(i) \end{aligned} \quad (22)$$

where $\xi^T(k) = [\mathbf{z}^T(k) \mathbf{z}^T(k+h_1-h(k))]$ and

$$\mathbf{\Omega} = \begin{bmatrix} -\mathbb{P} + (\tau+1)\mathbf{Q} + \tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}} + \tau \hat{\mathbf{A}}^T \mathbf{R} \hat{\mathbf{A}} & \tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}}_d + \tau \hat{\mathbf{A}}^T \mathbf{R} \tilde{\mathbf{A}}_d \\ * & \tilde{\mathbf{A}}_d^T \mathbf{P} \tilde{\mathbf{A}}_d - \mathbf{Q} + \tau \tilde{\mathbf{A}}_d^T \mathbf{R} \tilde{\mathbf{A}}_d \end{bmatrix}$$

Define $\varsigma^T(k) = [\mathbf{z}^T(k) \mathbf{z}^T(k+h_1-h(k)) \mathbf{y}^T(i)]$ and

$$\Xi = \begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} & \mathbb{Y} \\ * & \mathbb{X}_{22} & \mathbb{Z} \\ * & * & \mathbb{R} \end{bmatrix} \geq 0 \quad (23)$$

where Ξ is partitioned in accordance with $\varsigma(k)$, and

$$\mathbb{X} = \begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ * & \mathbb{X}_{22} \end{bmatrix} = \begin{bmatrix} \mathbb{Y} \\ \mathbb{Z} \end{bmatrix} \mathbb{R}^{-1} \begin{bmatrix} \mathbb{Y} \\ \mathbb{Z} \end{bmatrix}^T \quad (24)$$

By equation (10)

$$\begin{aligned} \sum_{i=k+h_1-h(k)}^{k-1} \varsigma^T(k) \Xi \varsigma(k) &\leq \mathbf{z}^T(k) (\tau \mathbb{X}_{11} + \mathbb{Y}^T + \mathbb{Y}) \mathbf{z}(k) \\ &\quad + 2 \mathbf{z}^T(k) (\tau \mathbb{X}_{12} - \mathbb{Y} + \mathbb{Z}^T) \mathbf{z}(k+h_1-h(k)) \\ &\quad + \mathbf{z}^T(k+h_1-h(k)) (\tau \mathbb{X}_{22} - \mathbb{Z} - \mathbb{Z}^T) \mathbf{z}(k+h_1-h(k)) \\ &\quad + \sum_{i=k+h_1-h(k)}^{k-1} \mathbf{y}^T(i) \mathbf{R} \mathbf{y}(i) \end{aligned} \quad (25)$$

From equations (22) and (25)

$$\begin{aligned} \Delta V(k) &\leq \xi^T(k) \mathbf{\Omega} \xi(k) - \sum_{i=k+h_1-h(k)}^{k-1} \mathbf{y}^T(i) \mathbf{R} \mathbf{y}(i) \\ &\quad + \sum_{i=k+h_1-h(k)}^{k-1} \varsigma^T(k) \Xi \varsigma(k) - \sum_{i=k+h_1-h(k)}^{k-1} \varsigma^T(k) \Xi \varsigma(k) \\ &= \xi^T(k) \mathbf{\Phi} \xi(k) + \tau \xi^T(k) \mathbb{X} \xi(k) \\ &\quad - \sum_{i=k+h_1-h(k)}^{k-1} \varsigma^T(k) \Xi \varsigma(k) \end{aligned} \quad (26)$$

where

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} + \tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}} + \tau \mathbf{Q} + \tau \hat{\mathbf{A}}^T \mathbf{R} \hat{\mathbf{A}} & \phi_{12} + \tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}}_d + \tau \hat{\mathbf{A}}^T \mathbf{R} \tilde{\mathbf{A}}_d \\ * & \phi_{22} + \tilde{\mathbf{A}}_d^T \mathbf{P} \tilde{\mathbf{A}}_d + \tau \tilde{\mathbf{A}}_d^T \mathbf{R} \tilde{\mathbf{A}}_d \end{bmatrix}$$

Clearly in $\mathbf{\Phi} + \tau \mathbb{X} < 0$ implies $\Delta V(k) < 0$ for all non-zero $\xi(k)$, and thus the asymptotic stability of the augmented system (7) and (5) with a time-varying delay $0 \leq h_1 \leq h(k) \leq h_2$. Furthermore, by equation (24) $\mathbf{\Phi} + \tau \mathbb{X} < 0$ is equivalent to equation (8), according to the Schur complement. This completes the proof.

Remark 1. Note in the proof of Theorem 1, no commonly used [20–22] inequalities, such as the simple inequality $-2\mathbf{a}^T \mathbf{b} \leq \mathbf{a}^T \mathbf{W}^{-1} \mathbf{a} + \mathbf{b}^T \mathbf{W} \mathbf{b}$ for all $n \times 1$ vectors $\mathbf{a}$, $\mathbf{b}$ and $n \times n$ matrix $\mathbf{W} > 0$ and some more complicated inequalities, are applied. This

reduces potential conservatism of the derived stability condition.

Remark 2. To determine the largest h_2 for the stability of equation (5), a straightforward but slow method is to check the feasibility of LMI (8) for various integer values of τ one by one until the largest τ is found. Alternatively the feasibility of equation (8) with τ as a variable can be cast into a generalized eigenvalue problem (GEVP)

$$\begin{bmatrix} Q + \hat{\mathbf{A}}^T \mathbf{R} \hat{\mathbf{A}} & \hat{\mathbf{A}}^T \mathbf{R} \tilde{\mathbf{A}}_d & \mathbf{Y} \\ * & \tilde{\mathbf{A}}_d^T \mathbf{R} \tilde{\mathbf{A}}_d & \mathbf{Z} \\ * & * & -\mathbf{R} \end{bmatrix} < \lambda \begin{bmatrix} \Psi & \mathbf{0} \\ * & \mathbf{0} \end{bmatrix} \quad (27)$$

where $\lambda = 1/\tau$ is a real variable

$$\Psi = \begin{bmatrix} -\phi_{11} - \tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}} & -\phi_{12} - \tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}}_d \\ * & -\phi_{22} - \tilde{\mathbf{A}}_d^T \mathbf{P} \tilde{\mathbf{A}}_d \end{bmatrix}$$

and the following optimization problem can be formed to search for the smallest λ

$$\min_{\lambda > 0, \mathbf{P} > 0, \mathbf{Q} > 0, \mathbf{R} > 0, \mathbf{Y}, \mathbf{Z}} \lambda \quad \text{subject to equation (27)} \quad (28)$$

However, to utilize existing software such as the LMI Control Toolbox of Matlab [28], some adjustment is needed to match the problem formulation therein. Let $\mathbf{T} > 0$ be an additional symmetric matrix variable of proper dimension, and change equation (27) to

$$\begin{bmatrix} Q + \hat{\mathbf{A}}^T \mathbf{R} \hat{\mathbf{A}} & \hat{\mathbf{A}}^T \mathbf{R} \tilde{\mathbf{A}}_d & \mathbf{Y} \\ * & \tilde{\mathbf{A}}_d^T \mathbf{R} \tilde{\mathbf{A}}_d & \mathbf{Z} \\ * & * & -\mathbf{R} \end{bmatrix} < \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ * & \mathbf{0} \end{bmatrix} \quad (29)$$

$$\mathbf{T} < \lambda \Psi \quad (30)$$

Note that equation (29) and equation (30) together are basically equivalent to equation (27), except now with $\mathbf{T} > 0$ as an additional condition, but due to $\mathbf{R} > 0$ and $\mathbf{Q} > 0$ in the left-hand side of equation (27), the condition is not supposed to cause too much conservativeness. Therefore, the following GEVP optimization problem is adopted as a substitute for equation (28)

$$\min_{\lambda > 0, \mathbf{P} > 0, \mathbf{Q} > 0, \mathbf{R} > 0, \mathbf{Y}, \mathbf{Z}, \mathbf{T} > 0} \lambda \quad \text{subject to equations (29)}$$

$$\text{and (30)} \quad (31)$$

When the time-varying delay $h(k)$ reduces to a constant h satisfying $0 \leq h_1 \leq h \leq h_2$, Theorem 1 can

be specialized to the following corollary by modifying the Lyapunov function $V(k)$ with $V_3(k)$ dismissed and $V_2(k)$ changed to $V_2(k) = \sum_{i=k+h_1-h_2}^{k-1} \mathbf{z}^T(i) \mathbf{Q} \mathbf{z}(i)$. The proof is a reduced version of that for Theorem 1.

Corollary 1. The system (5) with $h(k) = h$ is asymptotically stable for $0 \leq h_1 \leq h \leq h_2$, if there exist matrices $\mathbf{P} > 0, \mathbf{Q} > 0, \mathbf{R} > 0, \mathbf{Y}$, and $\mathbf{Z}$, all with the size $(h_1 + 1)n \times (h_1 + 1)n$, such that

$$\begin{bmatrix} \phi_{11} + \tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}} + \tau \hat{\mathbf{A}}^T \mathbf{R} \hat{\mathbf{A}} & \phi_{12} + \tilde{\mathbf{A}}^T \mathbf{P} \tilde{\mathbf{A}}_d + \tau \hat{\mathbf{A}}^T \mathbf{R} \tilde{\mathbf{A}}_d & \tau \mathbf{Y} \\ * & \phi_{22} + \tilde{\mathbf{A}}_d^T \mathbf{P} \tilde{\mathbf{A}}_d + \tau \tilde{\mathbf{A}}_d^T \mathbf{R} \tilde{\mathbf{A}}_d & \tau \mathbf{Z} \\ * & * & -\tau \mathbf{R} \end{bmatrix} < 0 \quad (32)$$

For the uncertain system (1) with a time-varying delay satisfying equation (2), the augmented system

$$\mathbf{z}(k+1) = \bar{\mathbf{A}}\mathbf{z}(k) + \bar{\mathbf{A}}_d\mathbf{z}(k+h_1-h(k)) \quad (33)$$

will be discussed, where

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} + \Delta\mathbf{A}(k) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \mathbf{0} \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{D} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{F}(k) [\mathbf{E}_a \quad \mathbf{0} \quad \cdots \quad \mathbf{0} \quad \mathbf{0}] = \tilde{\mathbf{A}} + \tilde{\mathbf{D}}\mathbf{F}(k)\tilde{\mathbf{E}}_a$$

$$\bar{\mathbf{A}}_d = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_d + \Delta\mathbf{A}_d(k) \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_d \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{D} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{F}(k) [\mathbf{0} \quad \mathbf{0} \quad \cdots \quad \mathbf{0} \quad \mathbf{E}_d]$$

$$= \tilde{\mathbf{A}}_d + \tilde{\mathbf{D}}\mathbf{F}(k)\tilde{\mathbf{E}}_d$$

The problem is still to find how large h_2 can be for equation (1) to remain asymptotically stable. By adapting Theorem 1 to the system (33) and with the help of Lemma 1, the following result holds.

Theorem 2. The system (1) with parametric uncertainties (3) and (4) is asymptotically stable for $0 \leq h_1 \leq h(k) \leq h_2$ if there exist matrices $\mathbb{P} > 0$, $\mathbb{Q} > 0$, $\mathbb{R} > 0$, $\mathbb{Y}$, and $\mathbb{Z}$, all with the size $(h_1 + 1)n \times (h_1 + 1)n$, such that

$$\begin{bmatrix} -\mathbb{P} & \mathbf{0} & \mathbf{0} & \mathbb{P}\tilde{\mathbf{A}} & \mathbb{P}\tilde{\mathbf{A}}_d & \mathbf{0} & \mathbb{P}\tilde{\mathbf{D}} \\ * & -\tau^{-1}\mathbb{R} & \mathbf{0} & \mathbb{R}\tilde{\mathbf{A}} & \mathbb{R}\tilde{\mathbf{A}}_d & \mathbf{0} & \mathbb{R}\tilde{\mathbf{D}} \\ * & * & -\mathbb{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & \phi_{11} + \tau\mathbb{Q} + \tilde{\mathbf{E}}_a^T \tilde{\mathbf{E}}_a & \phi_{12} + \tilde{\mathbf{E}}_a^T \tilde{\mathbf{E}}_d & \tau\mathbb{Y} & \mathbf{0} \\ * & * & * & * & \phi_{22} + \tilde{\mathbf{E}}_d^T \tilde{\mathbf{E}}_d & \tau\mathbb{Z} & \mathbf{0} \\ * & * & * & * & * & -\tau\mathbb{R} & \mathbf{0} \\ * & * & * & * & * & * & -\mathbf{I} \end{bmatrix} < 0 \quad (34)$$

Proof. By the Schur complement, equation (8) is equivalent to

$$\Sigma = \begin{bmatrix} -\mathbb{P} & \mathbf{0} & \mathbf{0} & \mathbb{P}\tilde{\mathbf{A}} & \mathbb{P}\tilde{\mathbf{A}}_d & \mathbf{0} \\ * & -\tau^{-1}\mathbb{R} & \mathbf{0} & \mathbb{R}\tilde{\mathbf{A}} & \mathbb{R}\tilde{\mathbf{A}}_d & \mathbf{0} \\ * & * & -\mathbb{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & \phi_{11} + \tau\mathbb{Q} & \phi_{12} & \tau\mathbb{Y} \\ * & * & * & * & \phi_{22} & \tau\mathbb{Z} \\ * & * & * & * & * & -\tau\mathbb{R} \end{bmatrix} < 0 \quad (35)$$

By replacing $\tilde{\mathbf{A}}$ in equation (35) with $\tilde{\mathbf{A}} = \tilde{\mathbf{A}} + \tilde{\mathbf{D}}\mathbf{F}(k)\tilde{\mathbf{E}}_a$ and $\tilde{\mathbf{A}}_d$ with $\tilde{\mathbf{A}}_d = \tilde{\mathbf{A}}_d + \tilde{\mathbf{D}}\mathbf{F}(k)\tilde{\mathbf{E}}_d$, a sufficient condition for the stability of the augmented system (33) with uncertainties is found to be

$$\Sigma + \begin{bmatrix} \mathbb{P}\tilde{\mathbf{D}} \\ \mathbb{R}\tilde{\mathbf{D}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{F}(k) \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{E}}_a & \tilde{\mathbf{E}}_d & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \tilde{\mathbf{E}}_a^T \\ \tilde{\mathbf{E}}_d^T \\ \mathbf{0} \end{bmatrix} \mathbf{F}^T(k) \begin{bmatrix} \tilde{\mathbf{D}}^T \mathbb{P} & \tilde{\mathbf{D}}^T \mathbb{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} < 0 \quad (36)$$

Table 1 Interval time delay stability of equation (5)

By Theorem 1	$0 \leq h_1 \leq h(k) \leq h_2$ $0 \leq h(k) \leq 10, 5 \leq h(k) \leq 11,$ $7 \leq h(k) \leq 12, 9 \leq h(k) \leq 13$
By Lemma 2 of [20]	$0 \leq h(k) \leq 8, 5 \leq h(k) \leq 11,$ $8 \leq h(k) \leq 12, 10 \leq h(k) \leq 13$
By Corollary 1 of [20]	$0 \leq h(k) \leq 8$
By Lemma 3 of [20]	$3 \leq h(k) \leq 10, 5 \leq h(k) \leq 11,$ $7 \leq h(k) \leq 12, 9 \leq h(k) \leq 13$

By Lemma 1, equation (36) is guaranteed for equation (4) if there exists a scalar $\varepsilon > 0$ such that

$$\varepsilon \Sigma + \varepsilon^2 \begin{bmatrix} \mathbb{P}\tilde{\mathbf{D}} \\ \mathbb{R}\tilde{\mathbf{D}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{D}}^T \mathbb{P} & \tilde{\mathbf{D}}^T \mathbb{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \tilde{\mathbf{E}}_a^T \\ \tilde{\mathbf{E}}_d^T \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{E}}_a & \tilde{\mathbf{E}}_d & \mathbf{0} \end{bmatrix} < 0 \quad (37)$$

With $\varepsilon\mathbb{P}$, $\varepsilon\mathbb{Q}$, $\varepsilon\mathbb{R}$, $\varepsilon\mathbb{Y}$, and $\varepsilon\mathbb{Z}$ replaced by $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Y}$, and $\mathbb{Z}$, respectively, it is seen that equation (37) is equivalent to equation (34) by the Schur complement. This completes the proof.

Table 2 Interval time delay stability of equation (1)

By Theorem 2	$0 \leq h_1 \leq h(k) \leq h_2$ $0 \leq h(k) \leq 5, 4 \leq h(k) \leq 6,$ $6 \leq h(k) \leq 7$
By Theorem 1 of [20]	$0 \leq h(k) \leq 4, 3 \leq h(k) \leq 5,$ $5 \leq h(k) \leq 6$
By Theorem 2 of [20]	$2 \leq h(k) \leq 5, 4 \leq h(k) \leq 6$
By Theorem 3 of [20]	$3 \leq h(k) \leq 5, 4 \leq h(k) \leq 6$

4 NUMERICAL EXAMPLES

Example 1 [20]. Consider the uncertain discrete-time system (1) with a time-varying delay (2) and

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix}, \mathbf{A}_d = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.02 \end{bmatrix},$$

$$\mathbf{E}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{E}_d = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Note that the nominal system (5) with the above $\mathbf{A}$ and $\mathbf{A}_d$ is asymptotically stable when the time delay is zero. For comparison purposes, several values of h_1 are found which make all eigenvalues of equation (7) with $h(k) = h_1$ be located inside the unit circle of the complex plane. To each h_1 the corresponding h_2 is determined to guarantee the asymptotic stability of the above system. By utilizing the LMI Control Toolbox of Matlab [28] to implement equation (31), several time delay intervals for the case without parametric uncertainties ($\mathbf{F}(k) = 0$) are computed and compared with those from Lemma 2, Corollary 1, and Lemma 3 of [20]. The comparison is shown in Table 1, where it is seen that less conservative results are obtained here. Then by applying Theorem 2 to the case with parametric uncertainties, several time delay intervals are computed and compared with those from Theorem 1, Theorem 2, and Theorem 3 of [20]. The comparison is shown in Table 2, and less conservative results are also obtained here. Finally, consider the nominal system (5) with a constant time delay h satisfying $0 \leq h_1 \leq h \leq h_2$. By setting $h_1 = 0$ and applying Corollary 1, a time delay range $0 \leq h \leq 16$ for asymptotic stability is obtained.

Example 2 [21]. Consider the nominal discrete-time system (5) with time-varying delay (2) and

$$\mathbf{A} = \begin{bmatrix} 0.6 & 0 \\ 0.35 & 0.7 \end{bmatrix}, \mathbf{A}_d = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.1 \end{bmatrix}$$

The system (5) with the above $\mathbf{A}$ and $\mathbf{A}_d$ is also asymptotically stable when the time delay is zero. For $h_1 = 2$, equation (31) gives an upper bound $h_2 = 15$ on the time delay to ensure the system stability, whereas Theorem 1 and Theorem 2 of [21] give $h_2 = 10$ and $h_2 = 13$, respectively. Therefore, Theorem 1 is less conservative for this example than the existing method.

Example 3. Consider the nominal discrete-time system (5) with time-varying delay (2) and

$$\mathbf{A} = \begin{bmatrix} 0.8 & -0.1 \\ 0.8 & 0.7 \end{bmatrix}, \mathbf{A}_d = \begin{bmatrix} 0.3 & 0 \\ 0.8 & 0.1 \end{bmatrix}$$

Note that the system (5) is unstable with the above $\mathbf{A}$ and $\mathbf{A}_d$ when the time delay is zero. However, for $h_1 = 5$, equation (31) gives a time delay interval $5 \leq h(k) \leq 6$, and for $h_1 = 6$ another interval $6 \leq h(k) \leq 7$ that guarantee the asymptotic stability. Furthermore, for the system with a time-invariant delay h satisfying $0 \leq h_1 \leq h \leq h_2$ and various h_1 , Corollary 1 generates the corresponding time delay intervals $3 \leq h \leq 4$, $4 \leq h \leq 6$, $5 \leq h \leq 8$, $6 \leq h \leq 10$, and $8 \leq h \leq 11$, which together imply the system is asymptotically stable for $3 \leq h \leq 11$.

5 CONCLUSIONS

Using the Lyapunov method and the LMI framework, this paper has proposed stability criteria for the asymptotic stability of uncertain and nominal unforced discrete-time systems with a time-varying state delay. Among other methods in the literature assuming the stability of time delay systems for zero delay, the basic assumption of this paper is the system is asymptotically stable for a possibly non-zero time delay. This makes the application more flexible and in some cases produces less conservative results. The numerical examples provided illustrate the application and efficacy of the developed results.

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