The generalized Cayley–Hamilton theorem for standard pencils

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Abstract: The Cayley–Hamilton theorem is extended from the case of one matrix to that of two matrices. The developed theorem is suitable for the standard form pencil $sE - A$. Let $\Delta(s) := \det(sE - A) = \sum_{r=0}^n a_r s^r$. Then we show that $\sum_{r=0}^n a_r A^r E^{n-r} = 0$. Such a neat result, which is very similar to the original theorem also, is due to some nice properties of the standard pair $\{E, A\}$. For example, E and A commute and share the same eigenspaces.

Keywords: Cayley–Hamilton theory; singular systems; standard pencils; characteristic equations; Weierstrass forms.

1. Introduction

The Cayley–Hamilton theorem is very famous in linear algebra. It says that every $n \times n$ constant matrix A satisfies its characteristic equation. Let

$$\begin{aligned}\alpha(s) &:= \det(sI - A) \\ &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0.\end{aligned}$$

Then

$$\begin{aligned}\alpha(A) &:= A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1A + \alpha_0I \\ &= 0.\end{aligned}$$

In this note, we will extend the theorem to the standard pencil $sE - A$ [7], where the matrix E may be singular. Let

$$\Delta(s) := \det(sE - A) = \sum_{r=0}^n a_r s^r$$

be the characteristic polynomial of $sE - A$. Then the generalized Cayley–Hamilton theorem will show

$$\Delta(E, A) := \sum_{r=0}^n a_r A^r E^{n-r} = 0.$$

The problem of the generalized Cayley–Hamilton theorem has been investigated by several researchers. In [3–6] this problem is attacked by the help of Leverrier’s algorithm. Since $\{E, A\}$ is not a standard pair in their discussions, the fact of $EA \neq AE$ makes the expression pretty complicated. It is difficult to imagine why their results can be regarded as the Cayley–Hamilton theorem. [7] has developed the concept of standard pencils. Nikoukhah et al. [7] have mentioned that

$$\{A^{n-1}, A^{n-2}E, \dots, E^{n-1}\}$$

spans the same subspace as

$$\{A^k E^l \mid k, l \text{ are nonnegative integers}\}.$$

The key equation in the generalized Cayley–Hamilton theorem, $\Delta(E, A) = 0$, is believed to be first proposed in this note.

2. Main results

Definition 1 [1]. Let $\hat{E}$ and $\hat{A}$ be two square constant matrices. If

$$\det(s\hat{E} - \hat{A}) \neq 0,$$

then $s\hat{E} - \hat{A}$ is a *regular pencil*.

Definition 2 [7]. Let $sE - A$ be a regular pencil. If there exist scalars α and β such that $\alpha E + \beta A = I$, then $sE - A$ is a *standard pencil*.

Note that the characteristic polynomial of A is $\det(sI - A)$. Here, $sI - A$ is a standard pencil since we can take $\alpha = 1$ and $\beta = 0$. Furthermore, any regular pencil $s\hat{E} - \hat{A}$ can be easily transformed into a standard pencil. We just let multiply $s\hat{E} - \hat{A}$ by $(\mu\hat{E} - \hat{A})^{-1}$, where μ is an arbitrary scalar satisfying

$$\det(\mu\hat{E} - \hat{A}) \neq 0.$$

A standard pencil, $sE - A$, will be

$$E := (\mu \hat{E} - \hat{A})^{-1} \hat{E}, \quad (1)$$

and

$$A := (\mu \hat{E} - \hat{A})^{-1} \hat{A}. \quad (2)$$

If we choose $\alpha = \mu$ and $\beta = -1$, then

$$\mu E - A = I. \quad (3)$$

Hence, it is clear that $sE - A$ is indeed a standard pencil. In the following, we assume that $\{E, A\}$ is an $n \times n$ standard pair satisfying (3) in which E may be singular. First of all, we will construct the Weierstrass form [1] of $sE - A$. Let U be the modal matrix of E . Then E can be similarly transformed into its Jordan form,

$$U^{-1}EU = \begin{bmatrix} E_s & 0 \\ 0 & E_f \end{bmatrix} =: \bar{E}, \quad (4)$$

where E_s is an $r \times r$ nonsingular matrix and E_f is an $(n-r) \times (n-r)$ nilpotent matrix. From (3), $A = \mu E - I$, and hence

$$\begin{aligned} U^{-1}AU &= \begin{bmatrix} \mu E_s - I & 0 \\ 0 & \mu E_f - I \end{bmatrix} \\ &= \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} =: \bar{A}, \end{aligned} \quad (5)$$

where the sizes of A_s and A_f are the same as those of E_s and E_f respectively. The standard pencil thus can be transformed into

$$\begin{aligned} U^{-1}(sE - A)U &= (s\bar{E} - \bar{A}) \\ &= \begin{bmatrix} sE_s - A_s & 0 \\ 0 & sE_f - A_f \end{bmatrix} \end{aligned} \quad (6)$$

in which $sE_s - A_s$ and $sE_f - A_f$ represent the slow and fast part of the Weierstrass form.

Once we have (6), the characteristic polynomial of $sE - A$ will be shown to be the same as that of $sE_s - A_s$. Let

$$\det(sE_s - A_s) = a_r s^r + a_{r-1} s^{r-1} + \cdots + a_1 s + a_0.$$

Then $a_r = \det(E_s) \neq 0$. Also,

$$\det(sE_f - A_f) = \det(sE_f - \mu E_f + I) = 1,$$

since E_f is a nilpotent matrix in Jordan form. So,

$$\begin{aligned} \det(sE - A) &= \det(s\bar{E} - \bar{A}) \\ &= \det(sE_s - A_s) \\ &= a_r s^r + a_{r-1} s^{r-1} + \cdots + a_1 s + a_0 \\ &= a_n s^n + a_{n-1} s^{n-1} + \cdots \\ &\quad + a_r s^r + \cdots + a_1 s + a_0 \end{aligned} \quad (7)$$

where $a_n = a_{n-1} = \cdots = a_{r+1} = 0$, and $a_r \neq 0$.

Some nice properties, such as E and A commuting, of a standard pair will be investigated.

Lemma 1. (i) $EA = AE$.

(ii) $\bar{E}\bar{A} = \bar{A}\bar{E}$.

(iii) $E_s A_s = A_s E_s$ and $E_f A_f = A_f E_f$.

(iv) for any nonnegative integers l, k ,

$$E^k A^l = U(\bar{E}^k \bar{A}^l)U^{-1} = U(\bar{A}^l \bar{E}^k)U^{-1}.$$

Proof. (i) By (3), $A = \mu E - I$. So,

$$EA = E(\mu E - I) = (\mu E - I)E = AE.$$

(ii) By (4) and (5),

$$\begin{aligned} \bar{E}\bar{A} &= (U^{-1}EU)(U^{-1}AU) = U^{-1}EAU \\ &= U^{-1}AEU = (U^{-1}AU)(U^{-1}EU) = \bar{A}\bar{E}. \end{aligned}$$

(iii) It is clear by the result of (ii) as well as (4) and (5).

(iv) It is clear by noting that $U^{-1}U = I$, $E = U\bar{E}U^{-1}$, $A = U\bar{A}U^{-1}$, and $\bar{E}\bar{A} = \bar{A}\bar{E}$. $\square$

The Cayley–Hamilton theorem of slow part pencil, $sE_s - A_s$, is described by the next lemma.

Lemma 2. If

$$\det(sE_s - A_s) = \sum_{i=0}^r a_i s^i,$$

then

$$\Delta(E_s, A_s) := \sum_{i=0}^r a_i A_s^i E^{r-i} = 0.$$

Proof. We have

$$\begin{aligned} \det(sE_s - A_s) &= \det(E_s) \det(sI - E_s^{-1}A_s) \\ &= a_r (s^r + \tilde{a}_{r-1} s^{r-1} + \cdots + \tilde{a}_1 s + \tilde{a}_0) \end{aligned}$$

where $a_r = \det(E_s) \neq 0$ and $\tilde{a}_i = a_i/a_r$ for $i = 1, 2, \dots, r-1$

By applying the Cayley–Hamilton theorem to $E_s^{-1}A_s$, we have

$$(E_s^{-1}A_s)^r + \tilde{a}_{r-1}(E_s^{-1}A_s)^{r-1} + \dots + \tilde{a}_1(E_s^{-1}A_s) + \tilde{a}_0I = 0.$$

If we left multiply the above equation with $a_r E_s^r$ and use the result of Lemma 1(iii), $E_s A_s = A_s E_s$, we have

$$a_r A_s^r + a_{r-1} A_s^{r-1} E_s + \dots + a_1 A_s E_s^{r-1} + a_0 E_s^r = 0. \quad \square$$

Now, the main theorem of the generalized Cayley–Hamilton theorem for the standard pencil is ready to be proved.

Theorem 1. Generalized Cayley–Hamilton Theorem. *The standard pair $\{E, A\}$ satisfies its own characteristic equation; that is, $\Delta(E, A) = 0$, where $\Delta(s) = \det(sE - A)$ is given in (7).*

Proof.

$$\begin{aligned} \Delta(E, A) &:= a_n A^n + a_{n-1} A^{n-1} E + \dots \\ &\quad + a_r A^r E^{n-r} + \dots + a_1 A E^{n-1} + a_0 E^n \\ &= (a_r A^r + a_{r-1} A^{r-1} E + \dots \\ &\quad + a_1 A E^{r-1} + a_0 E^r) E^{n-r} \\ &= \left\{ U(a_r \bar{A}^r + a_{r-1} \bar{A}^{r-1} \bar{E} + \dots \right. \\ &\quad \left. + a_1 \bar{A} \bar{E}^{r-1} + a_0 \bar{E}^r) U^{-1} \right\} \\ &\quad \cdot \{ U \bar{E}^{n-r} U^{-1} \} \\ &= U \left\{ a_r \begin{bmatrix} A_s^r & 0 \\ 0 & A_f^r \end{bmatrix} \right. \\ &\quad \left. + a_{r-1} \begin{bmatrix} A_s^{r-1} & 0 \\ 0 & A_f^{r-1} \end{bmatrix} \begin{bmatrix} E_s & 0 \\ 0 & E_f \end{bmatrix} + \dots \right. \\ &\quad \left. + a_1 \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} E_s^{r-1} & 0 \\ 0 & E_f^{r-1} \end{bmatrix} \right\} \end{aligned}$$

$$\begin{aligned} &+ a_0 \begin{bmatrix} E_s^r & 0 \\ 0 & E_f^r \end{bmatrix} \begin{bmatrix} E_s^{n-r} & 0 \\ 0 & E_f^{n-r} \end{bmatrix} U^{-1} \\ &= U \begin{bmatrix} \Delta(E_s, A_s) & 0 \\ 0 & \Delta(E_f, A_f) \end{bmatrix} \begin{bmatrix} E_s^{n-r} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} \\ &= U \begin{bmatrix} 0 & 0 \\ 0 & \Delta(E_f, A_f) \end{bmatrix} \begin{bmatrix} E_s^{n-r} & 0 \\ 0 & 0 \end{bmatrix} U^{-1} \\ &= 0. \end{aligned}$$

During the derivation, we have used $\Delta(E_s, A_s) = 0$ (Lemma 2) and $E_f^{n-r} = 0$ (E_f is an $(n-r) \times (n-r)$ nilpotent matrix). $\square$

3. Conclusion

We have developed the generalized Cayley–Hamilton theorem for standard pencils. Some nice properties of the standard pair $\{E, A\}$, such as E and A commuting, the eigenspaces of E and A being the same, etc. were used to simplify the mathematical expressions. The final result is very neat. One possible contribution of the generalized Cayley–Hamilton theorem seems to be in analyzing and designing solvable singular systems. It is well known that in order to guarantee the existence and the uniqueness of the solution of

$$\hat{E}\dot{x} = \hat{A}x + \hat{B}u,$$

$s\hat{E} - \hat{A}$ must be a regular pencil [8]. As mentioned in (1) and (2), if we left multiply $(\mu\hat{E} - \hat{A})^{-1}$ on both sides of the singular system equation, then we have

$$E\dot{x} = Ax + Bu,$$

where

$$E = (\mu\hat{E} - \hat{A})^{-1} \hat{E},$$

$$A = (\mu\hat{E} - \hat{A})^{-1} \hat{A},$$

and

$$B = (\mu\hat{E} - \hat{A})^{-1} \hat{B}.$$

Now $sE - A$ is a standard pencil, and hence the generalized Cayley–Hamilton theorem can be applied.

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