Fuzzy sliding mode controlled slider–crank mechanism using a PM synchronous servo motor drive

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Abstract

The dynamic response of a sliding mode controlled slider–crank mechanism, which is driven by a permanent magnet (PM) synchronous servo motor, is studied in this paper. The rod and crank are assumed to be rigid. The Hamilton’s principle and Lagrange multiplier method are applied to formulate the equation of motion. Reducing the differential-algebraic equations and employing the Runge–Kutta numerical method, the state variable representation is obtained. Moreover, based on the principles of the sliding mode control, a position controller is developed. Then, a simple fuzzy inference mechanism is utilized to estimate the upper bound of uncertainties for the sliding mode controller. Numerical results show that the dynamic behavior of the proposed controller–motor–mechanism system is robust to parametric variation and external disturbance. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords: fuzzy sliding mode control; slider–crank mechanism; servo motor

1. Introduction

The slider–crank mechanism is widely used. Examples of its application are found in gasoline and diesel engines, where the gas force acts on the slider and the motion is transmitted through the links. There are two extreme positions of the slider during one motion cycle of the crank. The steady-state solutions and the elastic stability of the motion of a slider–crank mechanism were obtained [1–3]. In addition, the response of the system is found by Viscomi and Arye [4] to be dependent upon the five parameters: length, mass, damping, external piston force and frequency. Fung [5] has described the dynamic analysis of a slider–crank mechanism with the flexible connecting rod which was considered as Timoskenko beam. However, in the previous studies, the

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dynamics of an electric motor was not considered and, moreover, the control effort was not applied to control the position, velocity or trajectory of the slider–crank mechanism.

In recent years, advancements in magnetic materials, semiconductor power devices, and control theory have made the PM synchronous servo motor drive plays a vitally important role in motion-control applications in the low-to-medium power range. The desirable features of the PM synchronous servo motor are its compact structure, high air-gap flux density, high power density, high torque-to-inertia ratio, and high torque capability. Moreover, compared with an induction servo motor, a PM synchronous servo motor has such advantages as higher efficiency, due to the absence of rotor losses and lower no-load current below the rated speed; and its decoupling control performance is much less sensitive to the parametric variation of the motor [6, 7]. To achieve fast four-quadrant operation and smooth starting and acceleration, the field-oriented control [8], or vector control, is used in the design of the PM synchronous servo motor drive. However, the control performance of the PM synchronous servo motor drive is still influenced by the uncertainties of the controlled plant, which usually comprise unpredictable plant parametric variation, external load disturbances, unmodelled and nonlinear dynamics.

In order to design a high-accuracy mechanical system, the full understanding of dynamic response of the mechanism and motor systems are very important. It is necessary to investigate the coupled system with an advanced control law in order to find the optimum operating conditions of the system, such as minimum angular velocity fluctuation and start-up time, which can lead to reduced vibration noise and stress, and increased machine life. This paper is interested in the formulation and dynamic behavior of a PM synchronous servo–motor coupled with a complexity mechanical system. In Figs. 1 and 2, a slider–crank mechanism system actuated by a PM synchronous servo motor and a geared speed-reducer is investigated. To control the coupled mechanical system with robust characteristics, a sliding mode controller [9, 10] is designed to control the position of the coupled mechanism. Furthermore, in the general sliding mode control, the upper bound of the uncertainties, which include parametric variation and external load disturbance, must be available. However, the bound of the uncertainties is difficult to obtain in advance for practical applications. A fuzzy sliding mode position controller is investigated to resolve the above difficulty, in which a simple PI-type fuzzy [11] inference mechanism is utilized to estimate the upper bound of uncertainties.
2. Dynamic analysis

2.1. The field-oriented PM synchronous motor drive

The machine model of a PM synchronous motor can be described in the rotor rotating reference frame as follows [12]:

\[ v_q = R_s i_q + p \lambda_d + \omega_s \dot{\lambda}_q, \]  
\[ v_d = R_s i_d + p \lambda_d - \omega_s \dot{\lambda}_q, \]

where

\[ \lambda_q = L_q i_q, \]  
and

\[ \lambda_d = L_d i_d + L_{md} I_{fd}. \]

In the above equations \( v_d \) and \( v_q \) are the \( d-, q- \)axis stator voltages, \( i_d \) and \( i_q \) are the \( d-, q- \)axis stator currents, \( L_d \) and \( L_q \) are the \( d-, q- \)axis inductances, \( \lambda_d \) and \( \lambda_q \) are the \( d-, q- \)axis stator flux linkages, while \( R_s \) and \( \omega_s \) are the stator resistance and inverter frequency respectively. The rotating reference \( d-q \) coordinates and the three-phase system are compared in Fig. 3. In Eq. (4) \( I_{fd} \) is the equivalent \( d- \)axis magnetizing current, and \( L_{md} \) is the \( d- \)axis mutual inductance.

The electric torque

\[ \tau_m = 3P \left[ L_{md} I_{fd} i_q + (L_d - L_q) i_d i_q \right] / 2, \]  
and the equation for the motor dynamics is

\[ \tau_e = \tau_m + B_m \omega_r + J_m \omega_r. \]

In Eq. (5) \( P \) is the number of pole pairs, \( \tau_m \) is the load torque, \( B_m \) is the damping coefficient, \( \omega_r \) is the rotor speed and \( J_m \) is the moment of inertia. The inverter frequency is related to the rotor speed as

\[ \omega_s = P \omega_r. \]
The basic principle in controlling a PM synchronous motor drive is based on field orientation. The flux position in the $d-q$ coordinates can be determined by the shaft-position sensor because the magnetic flux generated from the rotor’s permanent magnet is fixed in relation to the rotor shaft position. In Eqs. (4) and (5), if $i_d = 0$, the $d$-axis flux linkage $\lambda_d$ is fixed since $L_{md}$ and $I_{fd}$ are constant for a PM synchronous motor, and the electromagnetic torque $\tau_e$ is then proportional to $i_q$ which is determined by closed-loop control. The rotor flux is produced in the $d$-axis only, while the current vector is generated in the $q$-axis for the field-oriented control. Since the generated motor torque is linearly proportional to the $q$-axis current as the $d$-axis rotor flux is constant in Eq. (5), the maximum torque per ampere can be achieved.

The configuration of a general field-oriented PM synchronous motor drive system is shown in Fig. 4, which consists of a PM synchronous motor coupled with a mechanism, a ramp comparison current-controlled PWM voltage source inverter (VSI), a unit vector $(\cos \theta_s + j \sin \theta_s$, where $\theta_s$ is the position of rotor flux) generator, a coordinate translator, a speed control loop and a position control loop. The PM synchronous motor used in this drive system is a three-phase four-pole 750W 3.47 A 3000 rpm type.

With the implementation of field-oriented control, the PM synchronous motor drive system can be simplified to a control system block diagram, as shown in Fig. 5, in which

$$\tau_e = K_t i_q^*,$$  

$$K_t = \frac{3}{2} P L_{md} I_{fd},$$  

$$H_p(s) = \frac{1}{J_m s + B_m}.$$
Fig. 4. Configuration of a field-oriented PM synchronous servo motor drive system.

Fig. 5. Block diagram of a PM synchronous servo motor using the field-oriented mechanism with sliding mode controller.

where $K_t$ and $i_d^*$ are the torque constant and torque current command, respectively. The sliding mode position controller, also shown in Fig. 5, will be discussed in the following section. For the convenience of the controller design, the position and speed signals in the control loop are set at $1V = 50 \text{ rad}$ and $1V = 50 \text{ rad s}^{-1}$. 
Fig. 1 shows a PM synchronous motor system including a geared speed-reducer with a gear ratio

$$n = \frac{n_a}{n_b} = \frac{\tau}{\tau_m} = \frac{\omega_r}{\omega} = \frac{\dot{\omega}_r}{\dot{\theta}}.$$  \hspace{1cm} (11)

where \(n\), \(n_a\) and \(n_b\) are the gear ratio and gear numbers. Substituting Eqs. (8) and (11) into Eq. (6), the following applied torque can be obtained:

$$\tau = n(\tau_c - J_m\dot{\omega}_r - B_m\omega_r) = n(K_i^* - nJ_m\dot{\theta} - nB_m\dot{\theta}),$$ \hspace{1cm} (12)

where \(\tau\) is the torque applying in the angle \(\theta\). The parameters of the motor system are:

$$K_i = 0.6732 \text{ Nm A}^{-1}, \quad n = 1, \quad J_m = 1.32 \times 10^{-3} \text{ Nm s}^2 = 0.066 \text{ Nms rad V}^{-1}, \quad (13)$$

$$B_m = 5.78 \times 10^{-3} \text{ Nm s rad}^{-1} = 0.289 \text{ Nm V}^{-1}.$$  

### 2.2. Mathematical model of the coupled mechanism

The slider-crank mechanism driven by a PM synchronous servo motor is shown in Fig. 2. The Hamilton's principle and the Lagrange multiplier are used to derive the differential-algebraic equation for the slider--crank mechanism in Appendix A.

The constraint position, velocity and acceleration equations (A4)--(A6) must hold. By using Eqs. (A.4)--(A.6) and (A.19), the following equation in the matrix form is obtained:

$$\begin{bmatrix} M & \Phi_\psi^T \\ \Phi_\psi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} BU + D(\psi) - N(\psi, \dot{\psi}) \end{bmatrix}.$$ \hspace{1cm} (14)

This is a system of differential-algebraic equations and the matrices element can be found in Appendix A.

### 2.3. Reduced system of differential equations of motion

The differential algebraic equation of mechanism motion derived above are summarized in the matrix form of Eq. (14), and the constraint equation of Eq. (A.4). Implicit method has to be employed to solve the equation of the system.

Eqs. (14) and (A.4) may be reordered and partitioned, according to the decomposition of \(\psi = [\theta \phi]^T = [v^T u^T]^T\) which is the same as [13]. If the constraints are independent, the matrix \(\Phi_\psi\) has full row rank, and there is always at least one nonsingular submatrix \(\Phi_u\) of rank 2.

Gauss–Jordan reduction of the matrix \(\Phi_\psi\) with double pivoting defines a partitioning of \(\psi = [v^T, u^T]^T, u = [\phi]^T, v = [\theta]\) such that \(\Phi_u\) is the submatrix of \(\Phi_\psi\) whose columns correspond to elements \(u\) of \(\psi\) and \(\Phi_e\) is the submatrix of \(\Phi_\psi\) whose columns correspond to element \(v\) of \(\psi\). The elements of the vectors \(u, v\) and matrices \(\Phi_u, \Phi_e\) are detailed in the Appendix B. Thus, Eqs. (14) and
(A4) can be rewritten as

\[ M^v \dddot{v} + M^v \ddot{v} + \Phi_v^T \lambda = B^v U + D^v - N^v, \]
\[ M^u \dddot{u} + M^u \ddot{u} + \Phi_u^T \lambda = B^u U + D^u - N^u, \]  \hspace{1cm} (15)
\[ \Phi_u \ddot{u} + \Phi_v \ddot{v} = \gamma , \]

or in the matrix form as

\[ \hat{M}(v) \dddot{v} + \hat{N}(v, \dot{v}) = \hat{Q} U + \hat{D} , \]  \hspace{1cm} (16)

where

\[ \hat{M} = M^v - M^v \Phi_u^{-1} \Phi_v - \Phi_v^T (\Phi_u^{-1})^T [M^u - M^u \Phi_u^{-1} \Phi_v], \]
\[ \hat{N} = [N^v - \Phi_v^T (\Phi_u^{-1})^T N^u] + [M^u \Phi_u^{-1} - \Phi_v^T (\Phi_u^{-1}) M^u \Phi_u^{-1}] \gamma, \]
\[ \hat{Q} = B^v - \Phi_v^T (\Phi_u^{-1})^T B^u, \quad \hat{D} = D^v - \Phi_v^T (\Phi_u^{-1})^T D^u. \]

The result is a set of differential equations with only one independent generalized coordinate \( v \). The equation is an initial value problem and can be integrated by using the fourth order Runge–Kutta method.

2.4. State variable representation

Let \( X = [v, \dot{v}]^T \) be the state variable vector, one can rewrite Eq. (16) as the state space dynamics as

\[ \dot{X} = f(X) + G U(t) + d(t), \]  \hspace{1cm} (17)

where

\[ f(X) = \begin{bmatrix} \dot{v} \\ -\hat{M}^{-1} \hat{N} \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \hat{M}^{-1} \hat{Q} \end{bmatrix}, \quad d(t) = \begin{bmatrix} 0 \\ \hat{M}^{-1} \hat{D} \end{bmatrix}, \quad U(t) = [i^*_q]. \]

3. Design of the sliding mode controller

Consider the second-order nonlinear, single-input–single-output (SISO) motor-mechanism coupled system:

\[ \ddot{v}(t) = f(X; t) + G(X; t) U(t) + d(t), \]  \hspace{1cm} (18)

where

\[ f(X; t) = -\hat{M}^{-1} \hat{N}, \quad G(X; t) = \hat{M}^{-1} \hat{Q}, \quad d(t) = \hat{M}^{-1} \hat{D}, \]

and \( U(t) \) is the control input \( i^*_q \). It is assumed that the function \( f \) is not exactly known, but the extent of the imprecision \( \Delta f \) is bounded by a known continuous function \( F(X; t) \). Similarly, the control gain \( G(X; t) \) is not exactly known, but is of constant sign and known bounds, i.e.

\[ 0 < G_{\min} \leq G(X; t) \leq G_{\max} . \]  \hspace{1cm} (19)
In addition, disturbance $d(t)$ is unknown, but is bounded by a known continuous function $D(X; t)$. According to the above description, then

\[ |f - \hat{f}| \leq F(X; t), \tag{20a} \]

\[ \frac{1}{\alpha} \leq \frac{\hat{G}(X; t)}{G(X; t)} \leq \alpha, \tag{20b} \]

\[ |d| \leq D(X; t), \tag{20c} \]

where $\hat{f}$ and $\hat{G}$ are nominal values of $f$ and $G$, respectively, and

\[ \alpha = (G_{\text{max}}/G_{\text{min}})^{1/2}. \]

The control problem is to find a control law so that the state $X$ can track the desired trajectories $X_d$ in the presence of the uncertainties. Let the tracking error vector be

\[ e = X - X_d = [e, \dot{e}]^T, \tag{21} \]

where $X_d = [v_d, \dot{v}_d]^T$, and the second derivative of $v_d$ is assumed to be zero. Let us define a sliding surface $S(t)$ in the state space $R^2$ by the scalar function $s(X; t) = 0$, where

\[ s(X, t) = Ce + \dot{e}, \quad C > 0. \tag{22} \]

The initial condition is

\[ e(0) = 0. \tag{23} \]

The tracking problem mentioned above is to find a control law $U(t)$ so that the state $X$ remaining on the surface $s(X, t) = 0$ for all $t \geq 0$.

### 3.1. Sliding mode control law design

In the design of the sliding mode control system, first is to find the equivalent control law, $U_{eq}$, which will keep the state of the system on the sliding surface. The equivalent control law is found by the following equation:

\[ s|_U = u_{eq} = 0. \tag{24} \]

Assuming all uncertainties are zero, then

\[ \hat{f} + \hat{G}U_{eq} + C\dot{e} = 0. \tag{25} \]

Solving Eq. (25), one can obtain

\[ U_{eq} = (\hat{G})^{-1}\dot{U}, \tag{26} \]

where

\[ \dot{U} = -\hat{f} - C\dot{e}. \tag{27} \]
Thus, given \( \dot{s}(X; t) = 0 \), the dynamics of the system on the sliding surface for \( t \geq 0 \) is given by
\[
\dot{\nu}(t) = -C\dot{c}.
\] (28)

### 3.2. Hitting control law design

A Lyapunov function candidate is chosen as follows:
\[
V = \frac{1}{2} s^2(X; t).
\] (29)

It is shown below that, if there exists a positive constant \( \eta \), such that
\[
\dot{V} = \frac{1}{2} \frac{d}{dt} (s^2(X; t)) \leq -\eta |s|,
\] (30)

then the state trajectories will hit the sliding surface \( s \). In order to satisfy the hitting condition of Eq. (30) in the presence of uncertainties, the control law is chosen as follows:
\[
U = U_{eq} + U_n,
\] (31)

where
\[
U_n = - (\hat{G})^{-1} K \text{sgn}(s),
\] (32)

and
\[
\text{sgn}(s) = \begin{cases} 
1 & \text{if } s > 0, \\
-1 & \text{if } s < 0.
\end{cases}
\]

Then, Eq. (30) becomes
\[
ss \dot{s} = s(f + GU + d + C\dot{c}) \leq -\eta |s|.
\] (33)

Or equivalently,
\[
[f + d + C\dot{c}] \text{sgn}(s) + GU \text{sgn}(s) \leq -\eta.
\] (34)

Substituting Eqs. (26), (27), (31) and (32) into (34), the following equation is obtained:
\[
[f + \dot{\tau} + d] \text{sgn}(s) + \left(\frac{G}{\hat{G}} - 1\right) \dot{U} \text{sgn}(s) - \frac{G}{\hat{G}} K \leq -\eta.
\] (35)

The optimal value of \( K \) that satisfies Eq. (35) is
\[
K \geq \alpha(F + D + \eta) + (\alpha - 1)|\dot{U}|.
\] (36)

To alleviate the chattering phenomenon, the quasi-linear mode controller [14] which replaces the discontinuous control laws of Eq. (32) by a continuous control one inside a boundary layer
around the switching surface, is adopted. That is $U_n$ in Eq. (32) is replaced by

$$U_n = -(\hat{G})^{-1} K \text{sat}\left(\frac{s}{\varepsilon}\right),$$

where $\varepsilon > 0$ is the width of boundary, and the function of sat($s/\varepsilon$) is defined as

$$\text{sat}\left(\frac{s}{\varepsilon}\right) = \begin{cases} 1 & \text{if } s > \varepsilon, \\ \frac{s}{\varepsilon} & \text{if } -\varepsilon \leq s \leq \varepsilon, \\ -1 & \text{if } s < -\varepsilon. \end{cases}$$

This leads to tracking within a guaranteed precision $\varepsilon$ while allowing the alleviation of the chattering phenomenon.

4. Design of fuzzy sliding mode controller

In the general sliding mode control, the upper bound of uncertainties, which include parametric variation and external load disturbance, must be available. However, the bound of the uncertainties is difficult to obtain in advance for practical applications. Therefore, a fuzzy sliding mode controller is proposed here, in which a PI-type [11] fuzzy inference mechanism is used to estimate the upper bound of uncertainties. The fuzzy inference mechanism can construct the estimation model of uncertainties. Comparing with a conventional estimator, the fuzzy inference mechanism uses prior expert knowledge to accomplish control object more efficiently.

Replace $K$ by $K_f$ in Eq. (37), the following control law can be obtained:

$$U(t) = U_{eq} - (\hat{G})^{-1} K_f \text{sat}\left(\frac{s}{\varepsilon}\right),$$

where $K_f$ is estimated by the fuzzy inference mechanism.

Because the data manipulated in the fuzzy inference mechanism is based on fuzzy set theory, the associated fuzzy sets involved in the fuzzy control rules are defined as follows:

- $N$: Negative
- $Z$: Zero
- $P$: Positive
- $NH$: Negative Huge
- $NB$: Negative Big
- $NM$: Negative Medium
- $NS$: Negative Small
- $ZE$: Zero
- $PS$: Positive Small
- $PM$: Positive Medium
- $PB$: Positive Big
- $PH$: Positive Huge

and their universe of discourses are all assigned to be $[-6, 6]$. There is no universal method to determine their values. Therefore, trial and error is needed in practice. The significance of the universe of discourse is that the input variables belonging to respective universe of discourse are mapped from measured values with corresponding scaling factors. Moreover, the fuzzy inference mechanism will be simplified to build a decision look-up table by the mapping process. The membership functions for the fuzzy sets corresponding to switching surface $s$, $\dot{s}$ and the change of the upper bound of uncertainties $\Delta K_f$ are defined in Fig. 6.
Since only three fuzzy subsets, \( N, Z \) and \( P \), are defined for \( s \) and \( \dot{s} \), the fuzzy inference mechanism only contains nine rules. The resulting fuzzy inference rules are as follows:

**Rule 1:** If \( s \) is \( P \) and \( \dot{s} \) is \( P \) Then \( \Delta K_f \) is \( NH \)

**Rule 2:** If \( s \) is \( P \) and \( \dot{s} \) is \( Z \) Then \( \Delta K_f \) is \( NB \)

**Rule 3:** If \( s \) is \( P \) and \( \dot{s} \) is \( N \) Then \( \Delta K_f \) is \( NM \)

**Rule 4:** If \( s \) is \( Z \) and \( \dot{s} \) is \( P \) Then \( \Delta K_f \) is \( NS \)

**Rule 5:** If \( s \) is \( Z \) and \( \dot{s} \) is \( Z \) Then \( \Delta K_f \) is \( ZE \)

**Rule 6:** If \( s \) is \( Z \) and \( \dot{s} \) is \( N \) Then \( \Delta K_f \) is \( PS \)

**Rule 7:** If \( s \) is \( N \) and \( \dot{s} \) is \( P \) Then \( \Delta K_f \) is \( PM \)

**Rule 8:** If \( s \) is \( N \) and \( \dot{s} \) is \( Z \) Then \( \Delta K_f \) is \( PM \)

**Rule 9:** If \( s \) is \( N \) and \( \dot{s} \) is \( N \) Then \( \Delta K_f \) is \( PH \)

Rule 1 is the condition that the switching variable \( s \) is far away from the switching surface and the derivative of \( s \) is also positive, so a large \( \Delta K_f \) is required for the occurrence of the sliding mode condition. Moreover, Rule 5 implies that the switching variable \( s \) is on the switching surface and the derivative of \( s \) is also zero, so only very small \( \Delta K_f \) is required for the occurrence of the sliding mode condition. The same analysis can be used to explain the other fuzzy rules.

Fuzzy output \( \Delta K_f \) can be calculated by the center of area (COA) defuzzification as:

\[
\Delta K_f = \frac{\sum_{i=1}^{9} w_i c_i}{\sum_{i=1}^{9} w_i} = \left[ c_1 \cdots c_9 \right] \begin{bmatrix} w_1 \\ \vdots \\ w_9 \end{bmatrix} / \sum_{i=1}^{9} w_i = v^T W, \quad (40)
\]
where \( v = [c_1, \ldots, c_9] \) is the adjustable parameter vector; \( c_1 \) through \( c_9 \) are the center of the membership functions of \( \Delta K_f \); \( W = [w_1, \ldots, w_9] / \sum_{i=1}^{9} w_i \) is a firing strength vector. Then the upper bound of uncertainties can be obtained as follows:

\[
K_f(k) = K_f(k - 1) + \Delta K_f,
\]

where \( k \) is the number of iterations.

5. Numerical results

By using a Runge–Kutta fourth-order numerical integration method, Eq. (17) was solved for the motor–mechanism combination system. For numerical simulations, the parameters of the slider–crank mechanism are chosen as follows:

\[
m_1 = 3.64 \text{ kg}, \quad m_2 = 1.18 \text{ kg}, \quad m_3 = 1.8 \text{ kg}, \quad l = 0.305 \text{ m},
\]

\[
l' = 0.055 \text{ m}, \quad R = 0.12 \text{ m}, \quad r = 0.1 \text{ m}, \quad \mu = 0.1.
\]

In addition, the gains of the sliding mode control law are given by the following:

\[
C = 2, \quad \varepsilon = 0.01, \quad K = 200.
\]

Three simulation cases are addressed in the sliding mode control. In all the simulation cases, the control objective is to control the slider to move from the left to the right end. Hence, the initial angle of \( \theta \) is \( \pi \); the desired angle of \( \theta \) is equal to \( 2\pi \), and the stroke of the slider, \( \Delta X \), is equal to 0.2 m. First, the nominal case with external disturbance force \( F_E = 0 \) is considered. The responses of the crank angle, the slider position, the phase plane trajectory and the control input are shown in Fig. 7a–d,

![Fig. 7. Response trajectories of sliding mode control system (nominal case with \( F_E = 0 \) Nt). (a) crank angle; (b) slider position; (c) phase plane trajectory; (d) control input \( i_q^* \).](image-url)
respectively. It is clear that the control input current $i_q^*$ has two jerks, and one occurs in the startup time and the other happens as the system dynamic hitting the sliding line. Next, the parametric variation case with increasing mass of $m_3$ to 18 kg and external disturbance force $F_E = 0$ is considered, and the responses of the system are shown in Fig. 8a–d. It stands to reason that the transient responses are mostly the same as the Fig. 7a–c because the control input $i_q^*$ shown
in Fig. 8d is larger than that shown in Fig. 7d. Finally, Fig. 9a–d illustrate the nominal case with the external disturbance force $F_E = 50$ Nt. In Fig. 9d, it is obvious that the high control input $i^*_q$ provides high torque input to the mechanism during the transient period. According to the above results, the sliding mode control is robust to the presence of uncertainties.

Fig. 10. Response trajectories of fuzzy sliding mode system (nominal case with $F_E = 0$ Nt). (a) crank angle; (b) slider position; (c) phase plane trajectory; (d) control input $i^*_q$.

Fig. 11. Response trajectories of fuzzy sliding mode system (parametric variation case with $F_E = 0$ Nt). (a) crank angle; (b) slider position; (c) phase plane trajectory; (d) control input $i^*_q$. 
In addition, the fuzzy sliding mode control is also applied to control the system shown in Eq. (17). A PI-type fuzzy inference mechanism is used to estimate the upper bound of uncertainties in the nominal, parametric variation and applying external disturbance cases. First, the simulation results of the nominal case with external disturbance force $F_E = 0$ are given in Fig. 10a–d. From Fig. 10d, it is clear that the control input is less than the one of the sliding mode control during the transient period. Then, the simulation results for the parametric variation case with external disturbance force $F_E = 0$ (are shown in Fig. 11a–d. The robustness to parametric variation with small value of control input is clearly shown in Fig. 11, where the mass of the slider $m_3$ is changed to ten times a larger value. Fig. 12a–d show the simulation results of the nominal case with external disturbance force $F_E = 50$ Nt. The phase plane trajectory is shown in Fig. 12c, in which the sliding mode occurs once the system dynamic hitting the sliding line, and the robustness is kept by the fuzzy sliding mode controller. However, since the initial values of the bound of the uncertainties is zero for the fuzzy sliding mode controller, the responses of the fuzzy sliding mode controller is little sluggish compared with those of the sliding mode controller.

6. Conclusions

This paper successfully demonstrates the application of a sliding mode and a fuzzy sliding mode controller to the position control of a slider–crank mechanism using PM synchronous servo motor drive. The design procedure of the sliding mode controller and the proposed fuzzy sliding mode controller are described in detail. Moreover, three cases, i.e., the nominal, parametric variation and external disturbance, were simulated to test the effectiveness of the proposed controllers. The main
contributions of this paper are:
1. developing a complete mathematical model for the controller–motor–mechanism with PM synchronous servo motor, and
2. successfully using a fuzzy inference mechanism to estimate the upper bound of uncertainties for the sliding mode controller, and
3. robust control performances of the motor–mechanism coupled system being obtained by the proposed controllers to parametric variation and external disturbances.

Appendix A

In this appendix, Hamilton’s principle and Lagrange multiplier are used to derive the differential-algebraic equation for the slider–crank mechanism. The motor-mechanism coupled system is shown in Fig. 2. The slider–crank mechanism consists of three parts: crank, rod and slider. The positions of gravity centers of crank, rod and slider are, respectively,

\[ x_{1cg} = 0, \quad y_{1cg} = 0, \]  
\[ x_{2cg} = r \cos \theta + \frac{l \cos \phi}{2}, \quad y_{2cg} = \frac{l \sin \phi}{2}, \]  
\[ x_{3cg} = r \cos \theta + l \cos \phi + l', \quad y_{3cg} = 0. \]

The holonomic constraint equation [15] is

\[ \Phi(\psi) = r \sin \theta - l \sin \phi = 0, \]  
where \( \psi = [\theta \ \phi]^T \) is the vector of generalized coordinates.

The kinematic velocity and acceleration equations are obtained by taking the first and second derivatives of Eq. (A.4), respectively, as

\[ \Phi_\psi \dot{\psi} = 0, \]  
\[ \Phi_\psi \ddot{\psi} = - (\Phi_\psi \dot{\psi})_{\psi} \]  
where

\[ \Phi_\psi \dot{\psi} = r \dot{\theta} \cos \theta - l \dot{\phi} \cos \phi, \]  
\[ \Phi_\psi \ddot{\psi} = r \dot{\theta}^2 \sin \theta - l \dot{\phi}^2 \sin \phi. \]

Total kinetic energy is

\[ T = T_1 + T_2 + T_3, \]  
where \( T_1, T_2 \) and \( T_3 \) represent the kinetic energies of crank, rod and slider, respectively,

\[ T_1 = \frac{1}{2} m_1 R^2 \dot{\theta}^2, \]
Let $V_1$ and $V_2$ represent the potential energies of crank and rod, respectively,

$$
V_1 = 0,
$$

$$
V_2 = \frac{1}{2} m_2 gl \sin \phi.
$$

The virtual work $\delta W^A$ includes the applied torque $\tau$ with the virtual angle $\delta \theta$, the friction force $F_B$ and the external force $F_E$ with the virtual displacement $\delta X_B$. Thus, the following equation is obtained:

$$
\delta W^A = \tau \delta \theta + (F_B - F_E) \delta X_B = \tau \delta \theta + F_{BE} (-r \sin \theta \delta \theta - l \sin \phi \delta \phi),
$$

where

$$
F_B = - \mu m_3 g \text{sgn}(V_B),
$$

$$
\text{sgn}(V_B) = \begin{cases} 
1 & \text{if } V_B > 0, \\
0 & \text{if } V_B = 0, \\
-1 & \text{if } V_B < 0,
\end{cases}
$$

and $V_B$ is the velocity of the slider $B$.

Substituting Eq. (16) into Eq. (A.15) and rewriting in terms of the generalized coordinate $\psi$, then

$$
\delta W^A = - \delta \psi^T Q^A,
$$

where $Q^A$ is the generalized force and given as

$$
Q^A = \left[ \begin{array}{c} F_{BE} \sin \theta - n (K_q i_q^* - J_m \ddot{\theta}_r - B_m \dot{\theta}_r) \\ F_{BE} l \sin \phi \end{array} \right].
$$

The generalized constraint reaction force can be obtained in term of Lagrange multiplier (Haug, 1992) as

$$
Q^C = \Phi^T \lambda,
$$

where $\lambda$ is the Lagrange multiplier and

$$
\Phi = [ r \cos \theta \quad - l \cos \phi ].
$$

Thus, the virtual work by all constraint reaction forces is

$$
\delta W^C = \delta \psi^T Q^C.
$$

The general form of Hamilton’s principle is

$$
0 = \int_t^{t_2} [\delta L + \delta W^A + \delta W^C] dt
= \int_t^{t_2} \delta \psi^T \left[ \frac{\partial L}{\partial \dot{\psi}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) - Q^A + Q^C \right] dt,
$$

where $L = T - V$ is called the Lagrangian [15], and $V = V_1 + V_2$. 
Appendix B

The variational equations of Eq. (A.18) must hold for all $\delta \psi$. The varied path coincides with the true path at the two time ends $t_1$ and $t_2$. It follows that $\delta \psi(t_1) = \delta \psi(t_2) = 0$. Thus, the Euler–Lagrange equations of motion, accounting for both applied and constraint forces, are

$$M(\psi)\ddot{\psi} + N(\psi, \dot{\psi}) - BU - D(\psi) + \Phi_{\psi} = 0,$$

(A.19)

where

$$M(\psi) = \begin{bmatrix}
- \frac{1}{2}m_1R^2 - (m_2 + m_3)R^2 \sin^2 \theta - n^2 J_m & - (\frac{1}{2}m_2 - m_3)rl \sin \theta \sin \phi \\
- (\frac{1}{2}m_2 - m_3)rl \sin \theta \sin \phi & - \frac{1}{3}m_2l^2 - m_3l^2 \sin^2 \phi
\end{bmatrix},$$

$$N(\psi, \dot{\psi}) = \begin{bmatrix}
- (m_2 + m_3)R^2 \dot{\theta}^2 \sin \theta \cos \theta - (\frac{1}{2}m_2 + m_3)rl \dot{\psi}^2 \sin \theta \cos \phi - n^2 B_m \dot{\theta} \\
- (\frac{1}{2}m_2 + m_3)rl \dot{\theta}^2 \cos \theta \sin \phi - m_3l^2 \dot{\phi}^2 \sin \phi \cos \phi - \frac{1}{2}m_2gl \cos \phi
\end{bmatrix},$$

$$B = \begin{bmatrix}
- nK_t \\
0
\end{bmatrix}, \quad U = [l^*_q], \quad D(\psi) = \begin{bmatrix}
F_{BE}r \sin \theta \\
F_{BE}l \sin \phi
\end{bmatrix}.$$


