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First published on: 12 November 2009

To cite this Article Dai, Tian-Shyr and Lyuu, Yuh-Dauh (2009) 'Accurate approximation formulas for stock options with discrete dividends', Applied Economics Letters, 16: 16, 1657 — 1663, First published on: 12 November 2009 (iFirst)

To link to this Article: DOI: 10.1080/13504850701604078
URL: http://dx.doi.org/10.1080/13504850701604078

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Accurate approximation formulas for stock options with discrete dividends

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Pricing options on a stock that pays discrete dividends has not been satisfactorily settled in the literature. Frishling (2002) shows that there are three different models to model stock price with discrete dividends, but only one of these models is close to reality and generates consistent option prices. We follow Frishling (2002) by calling this model Model 3. Unfortunately, there is no analytical option pricing formula for Model 3, and many popular numerical methods such as trees are inefficient when used to implement Model 3. A new stock price model is proposed in this article. To guarantee that the option prices generated by this new model are close to those generated by Model 3, the distributions of the new model at exdividend dates and maturity approximate the distributions of Model 3 at those dates. To achieve this, a discrete dividend in Model 3 is replaced by a continuous dividend yield that can be represented as a function of discrete dividends and stock returns in the new model. Thus, the new model follows a lognormal diffusion process and the analytical option pricing formulas can be easily derived. Numerical experiments show that our analytical pricing formulas provide accurate pricing results.

I. Introduction

Pricing options on dividend-paying stocks is a long-standing question. By assuming that the stock price follows a lognormal diffusion, Black and Scholes (1973) arrive at their groundbreaking option pricing model for nondividend-paying stocks. Merton (1973) extends the model to the case, where the underlying stock pays a nonstochastic continuous dividend yield. He defines the cost of carrying of a stock as the risk-free interest rate less the dividend yield, and the stock is assumed to grow at the cost of the carrying rate. This continuous dividend yield assumption is widely adopted for pricing options as in Kraus (1985), Barone-Adesi and Whaley (1987), Broadie and Detemple (1995, 1996), Shackleton and Wojakowski (2001), Chang and Shackleton (2003) and many others. However, almost all stock dividends are paid discretely rather than continuously. We call this dividend setting the discrete dividend if

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the amounts of future dividends are assumed to be known today. Pricing options on a stock that pays discrete dividends seems to be investigated first in Black (1975).

The discrete-dividend option pricing problem has drawn a lot of attention in the literature. Three popular models for this problem are discussed in Frishling (2002), and these three models are briefly introduced as follows.

**Model 1:** Roll (1977) suggests that the stock price is divided into two parts: the stock price minus the present value of future dividends over the life of the option and the present value of future dividends. The former part (call it net-of-dividend stock price) is assumed to follow a lognormal diffusion process, whereas the latter part is assumed to grow at the risk-free rate. Vanilla options can be computed by applying the Black–Scholes formula with the stock price replaced by the net-of-dividend stock price. Cox and Rubinstein (1985) also call this model the ad hoc adjustment.

**Model 2:** Musiela and Rutkowski (1997), following Heath and Jarrow (1988), suggest that the cum-dividend stock price, defined as the stock price plus the forward values of the dividends paid from today up to maturity, follows a lognormal diffusion process. Thus, vanilla options can be computed by applying the Black–Scholes formula by replacing the stock price with the cum-dividend stock price and by adding the forward values of the dividends prior to maturity to the strike price.

**Model 3:** The stock price jumps down with the amount of dividend paid at the exdividend date, and follows lognormal price process between two exdividend dates.

Although the above models address the discrete-dividend option pricing problem, Frishling (2002) shows that they are incompatible with each other and generate very different prices with the same inputs. A brief sketch is given to show why Model 1 always generates lower option prices than Model 3. Assume that the volatility input to both models is $\sigma$. Model 1 sets the volatility of the net-of-dividend stock price as $\sigma$, whereas Model 3 sets the volatility of the stock price as $\sigma$. The volatility of the stock price in Model 1 is lower than that in Model 3 as the volatility of the present value of future dividends, a component of the stock price, is assumed to be zero in Model 1. Model 1, therefore, produces lower option prices, and the price difference between these two models becomes larger as $\sigma$ becomes larger. To remove this difference, Hull (2000) recommends that the volatility of the net-of-dividend stock price be adjusted by a simple formula. However, our article shows that the performance of Hull’s volatility adjustment is mixed. Similarly, we can also infer that Model 2 produces higher option prices than Model 3 as Model 2 assigns the volatility of the forward values of the dividends (which is not a part of stock price) to be $\sigma$.

The first two models are widely accepted in the academic literature (Geske, 1979; Whaley, 1981, 1982; Carr, 1998; Chance et al., 2002) partly because closed-form option pricing formulas can be easily derived. However, Frishling (2002) points out that only Model 3 can reflect the reality and provide more consistent option prices. His numerical results show that both Model 1 and Model 3 can produce unreasonable pricing results for American-style options and some exotic options. For example, he argues that Model 1 could incorrectly render a down-and-out barrier option worthless simply because the net-of-dividend stock price reaches the barrier when the dividends are large enough. In reality, the option has a reasonable chance to survive since these dividends are paid later than today. On the other hand, although Model 3 is much closer to reality than the other two models, it does not allow closed-form solutions for European-style option prices. Model 3 can be implemented by some numerical methods such as the tree method. But, a naïve application of the tree method results in a nonrecombining tree as in Fig. 1. Note that the tree size grows drastically with the number of exdividend dates. This unpleasant property renders the tree model inefficient.

![Fig. 1. A tree model for pricing stock options with discrete dividends](image-url)

**Notes:** A discrete dividend is paid out at time step 2. Three separate trees beginning at time step 2 are coloured in white, light gray and dark gray, respectively.
In addition to the first two models mentioned in Frishling (2002), efficient numerical algorithms and simple formulas can be constructed by approximating the discrete dividend with either (1) a fixed dividend yield on each exdividend date or (2) a fixed continuous dividend yield. Geske and Shastri (1985) construct a recombining tree by following the first approach. Although their tree model is efficient, numerical results in this article show that the pricing results can deviate significantly from the results of Model 3 in pricing European-style options. The second approach is followed by Chiras and Manaster (1978). They transform the discrete dividends into a fixed continuous dividend yield and then apply the Merton formula. As this approach can be shown to be equivalent to the first approach in pricing European-style options, it shares the same problem.

In this article, we will first construct a new stock price process (call it Model 4) that captures some important properties of Model 3. We then derive analytical pricing formulas for Model 4. To guarantee that the option prices generated by Model 4 are close to those generated by Model 3, the distributions of Model 4 at exdividend dates and maturity approximate the distributions of Model 3 at those dates. In fact, a discrete dividend paid at time \( t \) in Model 3 is replaced by a proper continuous dividend yield paid from the last exdividend date (or option initial date) to time \( t \) in Model 4. This continuous dividend yield is derived to be a function of discrete dividends and the stock returns by Taylor expansion to make the stock price (at exdividend date or at maturity) in Model 4 close to that in Model 3. The continuous dividend yield in Model 4 can be reinterpreted as the shift of the drift and the volatility of the stock return. Thus Model 4 follows the lognormal diffusion price process, and analytical option pricing formulas can be derived. Our approach can be easily extended to price an option with multiple discrete-dividend payouts. This property is useful as a stock can pay up to four dividends per annum in US, for example. Numerical results show that our pricing formulas can provide more accurate pricing results than other approximation methods mentioned above.

The article is organized as follows. The mathematical model is briefly covered in Section II. Model 4 and the corresponding pricing formulas are derived in Section III. We will first consider the single-discrete-dividend case and then extend our approach to the multiple-discrete-dividend case. Experimental results given in Section IV verify the accuracy of our pricing formulas. Section V concludes this article.

II. The Models

In Model 3, the stock price is assumed to follow the lognormal diffusion process in a risk-neutral economy:

\[
\frac{dS(t)}{S(t)} = rdt + \sigma dB(t)
\]

where \( S(t) \) denotes the stock price at time \( t \), \( r \) denotes the annual risk-free interest rate, \( \sigma \) denotes the volatility, and \( B(t) \) denotes the standard Brownian motion. Then the stock price \( S(t) \) can be represented as

\[
S(t) = S(s)e^{(r-0.5\sigma^2)(t-s)+\sigma(B(t)-B(s))}
\]

if no dividend is paid between time \( s \) and time \( t \). Assume that a discrete dividend \( D \) is paid at exdividend date \( \tau \). Then the stock price falls by the amount \( \alpha D \) at time \( \tau \). For simplicity, \( \alpha \) is assumed to be one in our pricing formulas. In general, \( \alpha \) can be less than 1 when considering the effect of tax on dividend income. An \( \alpha \neq 1 \) poses no difficulties for modifying our pricing formulas.

Assume that a stock option initiates at time 0 and matures at time \( T \). Then the payoff at time \( T \) is \( (S(T)-X)^+ \) for a vanilla call option and \( (X-S(T))^+ \) for a vanilla put option, where \( X \) denotes the strike price and \( (A)^+ \) denotes \( \max(A, 0) \). The underlying stock is assumed to pay \( n \) discrete dividends between time 0 and time \( T \), where \( n \) is a positive integer. The \( j \)-th dividend \( c_j \) is paid at time \( \sum_{i=1}^{j-1} t_i \), where \( t_j \) denotes the time span between the \( (j-1) \)-th exdividend date (for \( j > 1 \)) or time 0 (for \( j = 1 \)) and the \( j \)-th exdividend date.

III. Analytical Formulas

We will first construct the stock price process for Model 4 in the single-discrete-dividend case and then derive an analytical pricing formula. For convenience, the stock price in Model 4 at time \( t \) is denoted as \( S(t) \). We further assume that \( S(0) = S(0) \). Later, we will extend our work to the multiple-discrete-dividend case. Although our discussions focus on call options, extension to put options is straightforward.

A stock option with single discrete dividend

First, consider a stock that pays only one discrete dividend \( c_1 \) at time \( t_1 \) before maturity \( T \). In Model 3, the stock price at time \( t_1 \)
is \( S(t_1) = S(0)e^{\mu t_1 + \sigma(B(t_1) - B(0))} - c_1 \), where \( \mu \equiv r - 0.5\sigma^2 \).

Thus the stock price \( S(T) \) is expressed as

\[
S(T) = [S(0)e^{\mu t_1 + \sigma(B(t_1) - B(0))} - c_1]e^{d(T-t_1)+\sigma(B(T)-B(t_1))}
\]

As \( S(T) \) is no longer lognormally distributed, closed-form option pricing formulas become hard to come by.

The stock price process in Model 4 is designed to follow a lognormal price process. To achieve this, we first replace the discrete dividend \( c_1 \) paid at time \( t_1 \) by a properly chosen continuous dividend yield \( q_1 \) paid from time 0 to \( t_1 \) as follows:

\[
S(t_1) = S(0)e^{\mu t_1 + \sigma(B(t_1) - B(0))} - c_1 \equiv S(0)e^{(\mu - q_1)t_1 + \sigma(B(t_1) - B(0))}
\]

An approximation solution for \( q_1 \) is then derived to make

\[
S(0)e^{(\mu - q_1)t_1 + \sigma(B(t_1) - B(0))} \approx 1 - e^{-q_1 t_1 - \frac{C_1 e^{\mu t_1} - e^{-\sigma(B(t_1) - B(0))}}{S(0)}}
\]

The left-hand side of the above equation can be approximated by the first-order Taylor expansion as \( 1 - (1 - q_1 t_1) \), and the right-hand side is approximated by \( k_1(1 - \sigma(B(t_1) - B(0))) \), where \( k_1 = C_1 e^{\mu t_1} / S(0) \). Thus we have \( q_1 \approx k_1(1 - \sigma(B(t_1) - B(0))) / t_1 \).

Finally, \( S(T) \) can be approximated by \( S'(T) \) as follows:

\[
S(T) = [S(0)e^{d(T-t_1)+\sigma(B(T)-B(t_1))} - c_1]e^{d(T-t_1)+\sigma(B(T)-B(t_1))}
\]

\[
S'(T) = S(0)e^{(\mu - q_1)t_1 + \sigma(B(t_1) - B(0))}e^{(\mu t_1 + \sigma(B(T) - B(0)))}
\]

\[
S'(T) \approx [S(0)e^{(\mu - q_1)t_1 + \sigma(B(t_1) - B(0))} - c_1]e^{(\mu t_1 + \sigma(B(T) - B(t_1)))}
\]

Note that \( S'(T) \) follows the lognormal distribution.

Let \( \text{Var}(X) \) denote the variance of the random variable \( X \). Define \( \sigma_1 \) by

\[
\sigma_1 = \sqrt{\frac{\text{Var}[k_1\sigma(B(t_1) - B(0)) + \sigma(B(T) - B(0))]}{T}}
\]

Thus the discrete dividend \( c_1 \) paid at time \( t_1 \) in Model 3 is replaced by a continuous dividend yield \( q_1 \) that can be approximately interpreted as the shift of the drift of the stock return from \( \mu \) to \( \mu - k_1 / T \) and the volatility from \( \sigma \) to \( \sigma_1 \) in Model 4. The value for a vanilla call option can be calculated by the risk-neutral variation method as follows:

\[
e^{-rT}E[S'(T) - X]^+ = e^{-rT}E[S(0)e^{(\mu - k_1 / T)T + k_1\sigma(B(T) - B(0))} + \sigma(B(T) - B(0)) - X]^+
\]

\[
= S'(0)e^{d(t_1)} - Xe^{-rT}N(d_2)
\]

\[
where a = \alpha^2 - \sigma^2 / 2T - k_1 \text{, } d_1 = \ln(S'(0)/X - k_1 + (\mu + \sigma_1^2)T) / \sigma_1 \sqrt{T} \text{, and } d_2 = d_1 - \sigma_1 \sqrt{T}.
\]

The accuracy of above formula can be improved by expanding \( k_1 e^{\sigma(B(t_1) - B(0))} \) further as \( k_1(1 - \sigma(B(t_1) - B(0))) + \sigma_1^2(B(t_1) - B(0))^2 / 2 \). To make Model 4 follow a lognormal price process, \( q_1 \) is derived as follows:

\[
q_1 \approx \frac{k_1}{t_1} \left[ 1 - \sigma(B(t_1) - B(0)) + \frac{\sigma_1^2(B(t_1) - B(0))^2}{2} \right]
\]

\[
\approx \frac{k_1}{t_1} [1 - \sigma(B(t_1) - B(0))] + \delta_1
\]

where \( \delta_1 \equiv E(k_1 \sigma_1^2(B(t_1) - B(0))^2 / 2) = k_1 \sigma_1^2 t_1 / 2 \).

Thus, \( S'(T) \) can be derived as follows:

\[
S'(T) \approx S'(0)e^{(\mu - k_1 / T)T + \sigma(B(T) - B(0)) + k_1\sigma(B(t_1) - B(0)) - \delta_1}
\]

A more accurate formula for a call option is then obtained by substituting Equation 6 into Equation 3. The resulting pricing formula can be expressed in terms of Equation 4 with \( a \) and \( d_1 \) redefined as \( (\alpha^2 - \sigma^2) / 2T - k_1 - \delta_1 \), and \( \ln(S(0)/X - k_1 + (\mu + \sigma^2)T - \delta_1) / \sigma \sqrt{T} \), respectively. Numerical experiments in Section IV show that this formula generates accurate prices.

### Multiple discrete dividends

The aforementioned approach can be further extended to price a stock option with multiple discrete dividends. We will first consider the two-discrete-dividend case and then describe the generalized pricing formula for the multiple-discrete-dividend case without proof.

Assume that two discrete dividends \( c_1 \) (paid at time \( t_1 \)) and \( c_2 \) (paid at time \( t_1 + t_2 \)) are paid prior to time \( T \). We again replace the dividend \( c_2 \) paid at time \( t_1 + t_2 \) by a proper continuous dividend yield \( q_2 \) paid
Analytics for stock options with discrete dividends

from time \( t_1 \) to time \( t_1 + t_2 \) as follows:

\[
S(t_1)e^{\mu t_2+\sigma(B(t_1+t_2)-B(t_1))} = c_2 \\
\equiv S(t_1)e^{(\mu-q_2)t_2+\sigma(B(t_1+t_2)-B(t_1))} \\
\Rightarrow 1 - e^{-q_2t_2} = \frac{C_2 e^{-\mu t_2}}{S(t_1)} e^{-\sigma(B(t_1+t_2)-B(t_1))} 
\]  

(7)

By substituting Equation 1 into Equation 7 with \( q_1 \approx (k_1 - \sigma(B(t_1) - B(0))) + \delta_1) / t_1 \), \( q_2 \) can be derived as follows:

\[
q_1 \approx k_1 [1 + (k_1)\sigma(B(t_1) - B(0)) - \sigma(B(t_2) - B(t_1))] + \delta_2 / t_2 
\]  

(8)

where \( k_2 \equiv c_2 e^{-\mu(t_1+t_2)+k_1+b_1}/S(0) \) and \( \delta_2 \equiv k_2[(1 + k_1)\sigma^2 t_1 + \sigma^2 t_2]/2 \). Thus \( S(T) \) can be approximated by \( S'(T) \) as follows:

\[
S(T) = \left( S(t_1)e^{\mu t_2+\sigma(B(t_1+t_2)-B(t_1))} - c_2 \right) \\
\times e^{(\mu-q_2)t_2+\sigma(B(t_1+t_2)-B(t_1))} \\
= S(t_1)e^{\mu t_2+\sigma(B(t_1+t_2)-B(t_1))} \\
\times e^{(\mu-q_2)t_2+\sigma(B(t_1+t_2)-B(t_1))} \\
= S(0)e^{(\mu-q_1)t_1+\sigma(B(t_1)-B(0))}e^{\mu t_2+\sigma(B(t_1+t_2)-B(t_1))} \\
\times e^{(\mu-q_2)t_2+\sigma(B(t_1+t_2)-B(t_1))} \\
\approx S'(0)e^{(\mu-k_1-k_2+b_1+b_2)t_2} + (1+k_2)\sigma(B(t_1) - B(0)) + (1+k_2)\sigma(B(t_1+t_2) - B(t_1)) + \sigma(B(T) - B(t_1+t_2)) \\
\equiv S'(T) 
\]

(9)

where we substitute Equations 5 and 8 into Equation 9. Note that \( S'(T) \) follows the lognormal distribution. Define \( \sigma_2 \) by

\[
\sigma_2 = \sqrt{\text{Var}[(1+k_1+k_2+k_1k_2)\sigma(B(t_1) - B(0)) + (1+k_2)\sigma(B(t_1+t_2) - B(t_1)) + \sigma(B(T) - B(t_1+t_2))]} / T \\
= \sqrt{(1+k_1+k_2+k_1k_2)^2\sigma^2 t_1 + (1+k_2)^2\sigma^2 t_2 + \sigma^2(T - t_1 - t_2)} / T
\]

Again, the discrete dividends \( c_1 \) and \( c_2 \) in Model 3 are replaced by continuous dividend yields \( q_1 \) and \( q_2 \) that can be approximately interpreted as the shift of the drift of the stock return from \( \mu \) to \( \mu - k_1 + k_2 + b_1 + b_2 / T \) and the volatility from \( \sigma \) to \( \sigma_2 \) in Model 4.

The value for a vanilla call option can be calculated by the risk-neutral valuation method as follows:

\[
e^{-RT}E[S'(T) - X]^+ = S'(0)e^{(\sigma_2^2 - \sigma^2)t_2 / 2 + k_1 - k_2 - b_1 - b_2 N(d'_1) - \sigma_2 N(d'_2)} \\
- Xe^{-\sigma^2t_2 / 2}N(d'_2)
\]

where \( d'_1 = \ln(S'(0)/X - k_1 - k_2 + (\mu + \sigma_2^2)T - b_1 - b_2)/\sigma_2 \sqrt{T} \) and \( d'_2 = d'_1 / \sigma_2 \sqrt{T} \).

IV. Numerical Results

We compare Geske and Shastri’s fixed dividend yield model, Hull’s volatility adjustment model, and all the four discrete dividend models mentioned earlier in
this article. Geske and Shastri (1985) use fixed dividend yields to approximate discrete dividends. The fixed dividend yield is defined as the discrete dividend amount divided by the initial stock price. For example, the dividend yield is 5% if the initial stock price is 100 and the discrete dividend is 5. We use FDY to denote their approach. Model 1 generates lower option prices than Model 3 as argued before. To remove this difference, Hull (2000) recommends that the volatility of net-of-dividend stock price be adjusted by the volatility of the stock price multiplied by $S(0)/C_0D$, where $D$ denotes the present value of future dividends paid between time 0 to time $T$. We use Hull to denote Hull’s volatility adjustment approach. Besides, we use Model 1, ..., Model 4 to denote the option prices generated by Model 1, ..., Model 4, respectively. Hull denotes volatility adjustment approach of Hull (2000). Option prices that deviate from Model 3 by 0.3 are marked by asterisks.

V. Conclusions

Traditional models for pricing options on discrete-dividend-paying stocks either produce inconsistent pricing results or are inefficient. Our article constructs a new stock price model by replacing discrete

<table>
<thead>
<tr>
<th>X</th>
<th>FDY</th>
<th>Model 1</th>
<th>Hull</th>
<th>Model 2</th>
<th>Model 4</th>
<th>Model 3</th>
<th>FDY</th>
<th>Model 1</th>
<th>Hull</th>
<th>Model 2</th>
<th>Model 4</th>
<th>Model 3</th>
</tr>
</thead>
</table>

Notes: The initial stock price is 100, the risk-free rate is 3%, the time to maturity is 1 year, and a five-dollar-dividend is paid at year 0.6. The volatilities of the stock price are shown in the first row. The strike prices are listed in the first column. FDY denotes the fixed dividend yield approach of Geske and Shastri (1985). Model 1, ..., Model 4 denote the option prices generated by Model 1, ..., Model 4, respectively. Hull denotes volatility adjustment approach of Hull (2000). Option prices that deviate from Model 3 by 0.3 are marked by asterisks.

Table 2. Pricing a call option with two discrete dividends

<table>
<thead>
<tr>
<th>X</th>
<th>FDY</th>
<th>Model 1</th>
<th>Hull</th>
<th>Model 2</th>
<th>Model 4</th>
<th>Model 3</th>
<th>FDY</th>
<th>Model 1</th>
<th>Hull</th>
<th>Model 2</th>
<th>Model 4</th>
<th>Model 3</th>
</tr>
</thead>
</table>

Notes: The numerical settings are the same as those settings in Table 1 except that a 2.5-dollar-dividend is paid at year 0.4 and year 0.8. Option prices that deviate from Model 3 by 0.3 are marked by asterisks.
dividends with proper continuous dividend yields which can be viewed as functions of discrete dividends and stock returns. This model follows a lognormal diffusion process and analytical pricing formulas can be easily derived. Numerical results verify the superiority of our approach.

Acknowledgements
We thank Ren-Her Wang for useful suggestions.

The author was supported in part by NSC grant 94-2213-E-033-024 and NCTU research grant for financial engineering and risk management project.

The author was supported in part by NSC grant 95-2213-E-002.

References