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拉格拉奇極小子流形交點之光滑化(2/2)

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中文摘要:

我們順利解決本計劃研究之問題，證明2維或3維的浸入緊緻特殊拉
格拉奇子流形，若是只有孤立自交點，則一定是一系列嵌入特殊拉格
拉奇子流形的極限。這個結果在更高維時，一般是不對，但我們證明
在某些特殊情況，定理依然成立。這個研究成果，已經寫成學術文章，
投稿於雜誌審稿中。在此計劃中，我們不但學會了一些很重要的分析
方法，文章同時引起國內外這方面專家的注意及興趣，也繼續發現一
些需要進一步發展及探討的問題。

關鍵詞：緊緻、特殊拉格拉奇子流形、浸入、嵌入
Abstract

In this project, we proof the following theorems:

**Theorem** Every compact, connected, and immersed special Lagrangian submanifold, which has only isolated transversal self-intersection points in a compact 2 or 3 dimensional Calabi-Yau manifold, is the limit of a family of embedded special Lagrangian submanifolds.

**Remark:** The theorem is not expected to hold in general when the dimension $n$ is bigger than 3. Never the less, we show that if the tangent planes at the self-intersection point satisfy an angle condition, then the theorem holds for any dimension (as follows). One can also try to do the connected sum of two special Lagrangian submanifolds. However, it is easy to see that this will not work by simply counting the dimension of local deformations of a special Lagrangian submanifold.

**Theorem** Suppose that $L$ is a compact, connected, and immersed special Lagrangian submanifold in a compact $n$-dimensional Calabi-Yau manifold, $n > 3$. Moreover, assume that $L$ has only isolated transversal self-intersection of two sheets and the two tangent planes at each intersection point satisfy the angle condition $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$. Then $L$ is the limit of a family of embedded special Lagrangian submanifolds.

**Keywords:** special Lagrangian, Calabi-Yau manifold.
Embedded Special Lagrangian Submanifolds
in Calabi-Yau Manifolds

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In this paper, we will prove the following theorem:

**Theorem** Every compact, connected, and immersed special Lagrangian submanifold, which has only isolated transversal self-intersection points in a compact 2 or 3 dimensional Calabi-Yau manifold, is the limit of a family of embedded special Lagrangian submanifolds.

**Remark:** N.C. Leung points out that the theorem is not expected to hold in general when the dimension $n$ is bigger than 3. We will discuss this point on section 2. Never the less, we show that if the tangent planes at the self-intersection point satisfy an angle condition, then the theorem holds for any dimension (see Theorem 4). One can also try to do the connected sum of two special Lagrangian submanifolds. However, it is easy to see that this will not work by simply counting the dimension of local deformations of a special Lagrangian submanifold [9]. This is pointed out by R. Schoen.

The method in this paper can also be used to deform a special Lagrangian submanifold with singularities. This will be discussed in a future paper. To make our presentation less messy, the constant $C$ in the paper may change in different contexts. Its dependency will be specified whenever it is essential.

This work is independent and different from A. Butcher's in [2] (see also [3]), but the technique is similar. We have had some discussions after the work was finished. Both the author and A. Butcher benefited from the discussions and made some changes in each other's work. We basically use the same setting as Butcher's in [2] and omit some computation to avoid repetition. However, there are still a few differences in the treatment. Some is due to the nature of the problem and some is for clarity and correctness. I also should point out that in [3], A. Butcher only considers $n \geq 3$. However,
it is easy to see that the results quoted work for $n = 2$ and there is no such restriction in [2]. When the arguments depend on the dimension, I will indicate clearly and discuss separately.

The author would like to thank A. Butcher, R. Schoen, and J. Wolfson for their useful discussions and interests in this work. She also likes to thank N.C. Leung's comments and explaining her the reasons. During the period of this research, the author ever visited the National Center for Theoretical Sciences in Taiwan and Tom Wan in Chinese University, HongKong. She wishes to thank their hospitality and organizing the stimulating mathematical activities. Finally, she would like to thank the support of the National Science Council of Taiwan. The research is partially supported by NSC 89-2115-M-002-018 and 90-2115-M-002-006.

1 Preliminaries

Calibrated geometry and the notion of special Lagrangian submanifold were developed by R. Harvey and H. B. Lawson in [6]. We refer to their paper for a detailed discussion of this subject. The followings are some basic definitions:

**Definition 1** A closed, differential $p$-form $\phi$ on a Riemannian manifold $N$ is called a calibration if its comass is 1. That is, $\phi(e_1, \ldots, e_p) \leq 1$ for any oriented, orthonormal $p$-frame on $TN$ and the equality holds at some place.

**Definition 2** A submanifold $M$ of $N$ is calibrated by $\phi$, if $\phi|_M = dV_M$, where $dV_M$ is the induced volume form on $M$.

A very useful property of calibrated submanifolds is illustrated in the next proposition.
**Proposition 1** [6] If $M$ is calibrated by $\phi$, then $M$ has the least volume among all representatives in its homology class.

For instance, a $p$-dimensional complex submanifold in a Kähler manifold $N$ is calibrated by $\frac{1}{p!} \omega^p$, where $\omega$ is the Kähler form on $N$, and hence is volume minimizing. R. Harvey and H. B. Lawson showed that $\text{Re } dZ$ in $\mathbb{R}^{2n}$, where $dZ = dz_1 \wedge \cdots \wedge dz_n$, is a calibration. The corresponding calibrated submanifolds are called special Lagrangian. The form $\text{Re } (e^{i\theta}dZ)$, where $\theta$ is a constant, is also a calibration, and its corresponding calibrated submanifolds are called special Lagrangian of phase $\theta$. In a Calabi-Yau manifold $N$, there exists a parallel holomorphic $(n, 0)$ form $\Omega$ which is of unit length. The $n$-form $\text{Re } \Omega$ is a calibration and a Lagrangian submanifold in $N$ is called special Lagrangian if it is calibrated by $\text{Re } \Omega$. Recall that a Lagrangian submanifold is a real $n$-dimensional submanifold on which the restriction of $\omega$ vanishes, where $2n$ is the real dimension of $N$.

G. Lawlor [8] modified an example of R. Harvey and H. B. Lawson [6] and defined the following submanifolds, which will be called Lawlor necks in this paper:

Assume that $a_1, \ldots, a_n$, $n \geq 2$, are $n$ positive real numbers and $a = (a_1, \ldots, a_n)$. Set

$$\theta_k(a, \mu) = \int_0^\mu \frac{a_k ds}{(1 + a_k s^2) \sqrt{P(s)}} \text{ for } \mu \geq 0,$$

where

$$P(s) = \frac{(1 + a_1 s^2) \cdots (1 + a_n s^2) - 1}{s^2}.$$
One can extend $\theta_k(a, \mu)$ to negative $\mu$ by $\theta_k(a, -\mu) = -\theta_k(a, \mu)$. Define $\Phi_a : R \times S^{n-1} \rightarrow R^{2n}$ by

$$\Phi_a(\mu, x_1, \cdots, x_n) = \{z_1x_1, \cdots, z_nx_n\},$$

where

$$x_1^2 + \cdots + x_n^2 = 1 \quad \text{and} \quad z_k = \frac{1}{a_k} + \mu^2 e^{i\theta_k(a, \mu)}.$$

Note that

$$\Phi_{\frac{\mu}{t}}(\mu, x_1, \cdots, x_n) = t\Phi_a(\frac{\mu}{t}, x_1, \cdots, x_n) \quad \text{for} \quad t > 0.$$

Hence we can assume $\inf_{k=1, \cdots, n} a_k = 1$. Denote

$$\theta_k(a) = \int_0^\infty \frac{a_k ds}{(1 + a_k s^2)^{1/2} P(s)}, \quad \text{for} \quad k = 1, \cdots, n.$$

One can prove that $\theta_1(a) + \cdots + \theta_n(a) = \frac{\pi}{2}$. By an argument in [8], there is a bijection between positive $\theta_1, \cdots, \theta_n$ satisfying $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$ and $a_1, \cdots, a_n$ satisfying $\inf_{k=1, \cdots, n} a_k = 1$. Moreover, G. Lawlor proved that the image of $\Phi_a$, which is denoted by $M_a$, is embedded, calibrated by $Im dZ$, and asymptotic to $P_\theta$ and $P_{-\theta}$, where $P_\theta$ is the plane

$$P_\theta = \{(t_1 e^{i\theta_1(a)}, \cdots, t_n e^{i\theta_n(a)}) : t_j \in R, \quad j = 1, \cdots, n\}.$$

Note that $M_a$, $P_\theta$ and $P_{-\theta}$ are special Lagrangian of phase $\frac{\pi}{2}$. By moving these spaces by a phase, we can always make them special Lagrangian. We thus will not specify the phase any more. But when we talk about special Lagrangian submanifolds in this paper, we do mean that they are calibrated by the same form, i.e. they are of the same phase. (see [2], [5], [7], [8]).

A. Butcher [2] studies carefully the asymptotic behavior of the above Lawlor neck. We summarize some of his results here for completeness. He proves
that $|\theta_k(a, \mu) - \theta_k(a)| \leq \frac{1}{n|\mu|}$. Moreover, there exists a positive real number $R_0$ so that $M_a \setminus B_{R_0}(0)$ can be written as the graph of the gradient of a function

$$\Psi : P_i \setminus B_{R_0}(0) \to R, \; i = 1, 2.$$ 

Here we split $\mathbb{R}^{2n}$ as $P_i \times P_i^\perp$ to write the graph. The function $\Psi$ has the properties that

$$|\Psi(x)| \leq \frac{C}{|x|^{n-2}}, \quad |\nabla \Psi(x)| \leq \frac{C}{|x|^{n-1}}, \quad |\nabla^2 \Psi(x)| \leq \frac{C}{|x|^{n}},$$

$$|\nabla^3 \Psi(x)| \leq \frac{C}{|x|^{n+1}}, \quad \text{and} \quad |\nabla^4 \Psi(x)| \leq \frac{C}{|x|^{n+2}}$$

for $x \in P_i$ with $|x| \geq R_0$. The constant $C$ depends only on $a$ and $n$. The scaled manifold

$$\varepsilon(M_a \setminus B_{R_0}(0)) = \varepsilon M_a \setminus B_{\varepsilon R_0}(0), \; \varepsilon > 0,$$

is the graph of the gradient of a function

$$\Psi_\varepsilon : P_i \setminus B_{\varepsilon R_0}(0) \to R, \; i = 1, 2.$$ 

The function $\Psi_\varepsilon(x) = \varepsilon^2 \Psi(\frac{x}{\varepsilon})$ satisfies

$$|\Psi_\varepsilon(x)| \leq \frac{C \varepsilon^n}{|x|^{n-2}}, \quad |\nabla \Psi_\varepsilon(x)| \leq \frac{C \varepsilon^n}{|x|^{n-1}}, \quad |\nabla^2 \Psi_\varepsilon(x)| \leq \frac{C \varepsilon^n}{|x|^{n}},$$

$$|\nabla^3 \Psi_\varepsilon(x)| \leq \frac{C \varepsilon^n}{|x|^{n+1}}, \quad \text{and} \quad |\nabla^4 \Psi_\varepsilon(x)| \leq \frac{C \varepsilon^n}{|x|^{n+2}}$$

for $x \in P_i$ with $|x| \geq \varepsilon R_0$.

\section{Local model}

Assume that $p$ is a self-intersection point and locally it is the transversal intersection of two sheets. We would like to use the Lawlor neck as a local
model. So we first need to find a Lawlor neck which is asymptotic to the
two tangent planes at \( p \). Then cut off a small ball at \( p \) on each sheet, glue
in a scaled Lawlor neck, and connect it to the original submanifold outside
the balls. However, there is a condition \( \theta_1 + \cdots + \theta_n = \frac{\pi}{2} \) for the planes
which the Lawlor neck can be asymptotic to. In this section, we will discuss
how the condition affects the application. In particular, we show that this
condition is always satisfied for our situation in dimension 2 and 3, but it is
not true when \( n \geq 4 \). Hence when \( n \geq 4 \), we need to add the angle condition
\( \theta_1 + \cdots + \theta_n = \frac{\pi}{2} \) in Theorem 4. We will also discuss why the assertion cannot
hold in general if \( n \geq 4 \).

Recall that a Lagrangian plane (which is always assumed to contain the
origin) in \( \mathbb{R}^{2n} \) is the image of the real \( x_1, \ldots, x_n \) plane by a linear transfor-
mation \( A \in U(n) \). Thus the set of Lagrangian planes can be identified with
\( U(n)/SO(n) \) [6]. Given a pair of Lagrangian planes \( P_1 \) and \( P_2 \), we claim
that in suitable coordinates, one can make \( P_1 \) to be the \( x_1, \ldots, x_n \) plane and
\( P_2 \) to be of the form \( \{ (t_1 e^{i\theta_1}, \ldots, t_n e^{i\theta_n}) : t_j \in \mathbb{R}, j = 1, \ldots, n \} \). This is
because the Lie algebra \( u(n) \) of \( U(n) \) is decomposed into the direct sum of
\( S \) and \( so(n) \), where \( S \) is the set of pure imaginary symmetric matrices and
\( so(n) \) is the set of real skew symmetric matrices. The subalgebra \( S \) and
\( so(n) \) corresponds to the \( -1 \) eigenspace and \( 1 \) eigenspace of the involution
\( \tau : u(n) \to u(n) \) respectively, where \( \tau(y) = -y^t \). Since one can diagonalize a
real symmetric matrix, it follows that \( S = \bigcup k T k^{-1} \), where \( T \) is a pure imagi-
inary diagonal matrix and \( k \) is in \( SO(n) \). The symmetric space \( U(n)/SO(n) \)
is exactly \( \exp S \). The claim is thus proved. We like to thank C.L. Terng’s
discussion on this observation. Furthermore, if we denote \( |\omega_j| \) by \( \beta_j \), we can
assume that

\[ 0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \frac{\pi}{2} \quad \text{and} \quad \beta_{n-1} \leq \beta_n \leq \pi - \beta_{n-1}. \]

They are exactly the characterizing angles between \( P_1 \) and \( P_2 \) as defined in [8]. Note that one has \( 0 \leq \sum_{j=1}^{n} \beta_j \leq \frac{n\pi}{2} \). Now suppose that \( P_1 \) and \( P_2 \) are two special Lagrangian planes which intersect only at the origin. Then \( \beta_1 > 0 \) and when \( n = 2 \) or \( 3 \), one has \( \sum_{j=1}^{n} \omega_j = 0 \). It implies that \( \beta_1 = \beta_2 \) in the case \( n = 2 \) and \( \beta_1 + \beta_2 = \beta_3 \) in the case \( n = 3 \). If we change the orientation on \( P_2 \), which is denoted by \(-P_2\), then its characterizing angles with \( P_1 \) satisfy \( \beta_1 + \beta_2 + \beta_3 = \pi \) in the case \( n = 3 \). Change the coordinates such that \( P_1 = P_0 \) and \(-P_2 = P_{-\theta} \), where \( \theta = (\frac{\beta_1}{2}, \frac{\beta_2}{2}, \frac{\beta_3}{2}) \). We thus can find a Lawlor neck which is asymptotic to \( P_1 \) and \(-P_2 \). When \( n = 2 \), one can also obtain the same conclusion. But this is not true when \( n \geq 4 \). For example, the \( x_1, \ldots, x_4 \) plane and \( y_1, \ldots, y_4 \) plane in \( \mathbb{R}^8 \) are both calibrated by \( Re \, dZ \) and contain the origin. However, all the characterizing angles between these two planes are \( \frac{\pi}{2} \). Hence the sum of the angles is \( 2\pi \) and there does not exist a Lawlor neck which is asymptotic to the \( x_1, \ldots, x_4 \) plane and \( y_1, \ldots, y_4 \) plane.

The geometric obstruction for finding a Lawlor neck in \( n \geq 4 \) comes from the followings: There is an angle criterion which says that the nonzero sum (oriented union) \( P_1 + P_2 \) is area minimizing if and only if the characterizing angles between them satisfy the inequality \( \beta_n \leq \beta_1 + \cdots + \beta_{n-1} \). (See [5], [8], [11].) Suppose that \( P_1 \) and \( P_2 \) are two special Lagrangian planes. By the property of calibration, we know that \( P_1 + P_2 \) is area minimizing. Assume that their characterizing angles satisfy \( \beta_n < \beta_1 + \cdots + \beta_{n-1} \) and there exists a special Lagrangian \( L \) asymptotic to \( P_1 \) and \( P_2 \). A Lawlor neck has the property that it is the union of compact hypersurfaces in a family of Lagrangian
planes. Assume that $L$ has the same property. We first find two Lagrangian planes $P_1'$ and $P_2'$ near $P_1$ and $P_2$, which are not special Lagrangian and whose characterizing angles $\{\beta_j\}$, $j = 1, \cdots, n$, still satisfy $\beta_n' < \beta_1' + \cdots + \beta_{n-1}'$. If the intersection of $L$ and $P_1' + P_2'$ is a compact hypersurface in $P_1' + P_2'$, then the intersection will be the boundary of a compact portion of $L$ and also be the boundary of a compact subset $E_1 + E_2$ in $P_1' + P_2'$. By the special Lagrangian condition on $L$ and applying the angle criterion to $P_1'$ and $P_2'$, we know that both sets are volume minimizing with the same boundary. It follows that they are calibrated by the same form, which is a contradiction because $P_1'$ and $P_2'$ are chosen to be not special Lagrangian. Thus we cannot have a Lawlor neck to approximate such a pair. Can we find local models of different nature to resolve the isolated self-intersection point in general? The answer is very likely still no. This is observed by N.C. Leung and the reason will be explained in next paragraph.

One can consider complex Lagrangian submanifolds in a hyperkähler manifold. Recall that there is a $S^2$ family of compatible complex structures in a hyperkähler manifold. A complex Lagrangian submanifold is a complex submanifold with respect to one of the compatible complex structures, and is special Lagrangian with respect to another compatible complex structure. By the property of calibration, any subspace (even singular) which presents the homology class of a complex Lagrangian submanifold and is volume minimizing in the class, must be calibrated by the same form, and hence also be complex Lagrangian. Thus all special Lagrangian submanifolds in the homology class of a complex Lagrangian submanifold are complex Lagrangian. It means that we must do the connected sum in the complex category. This is known to be impossible in general when the complex dimension is bigger
than one. In particular, when we add a handle \(\cong S^{n-1} \times R\) to the original submanifold, it will increase the dimension of the first homology group by one. If the original submanifold is complex Lagrangian, then this new topology cannot be complex Lagrangian because it does not satisfy a necessary condition of a Kähler manifold (the first homology group is even dimension).

The upshot for the above observation is that either the theorem does not hold in general when \(n > 3\), or one cannot find a compact, connected, complex Lagrangian submanifold in a hyperkähler manifold which has only isolated transversal self-intersection points.

3. Approximate submanifolds

Suppose that \(L\) is a compact, connected, and immersed special Lagrangian which has only isolated transversal self-intersection points in a compact \(n\)-dimensional Calabi-Yau manifold \(N\), where \(n \geq 2\). Without loss of generality, we can assume that there is only one self-intersection point \(p\) on \(L\) and locally it is the transversal intersection of two sheets of \(L\). In a small neighborhood of \(p\), the metric in \(N\) is equivalent to the Euclidean metric in \(R^{2n}\). Thus for simplicity, the distance and norm in the following construction of approximate submanifolds in this section are all with respect to the Euclidean metric unless specified explicitly. Assume that the ball of radius \(r_0\) at \(p\) in \(N\), which is denoted by \(B_{r_0}\), is both a Darboux neighborhood and a normal neighborhood near \(p\). That is, we can choose coordinates \(x_1, \ldots, x_n, y_1, \ldots, y_n\) such that \(p\) is the origin and for \(q \in B_{r_0}\):

1. the Kähler form satisfies \(\omega(q) = \sum_{i=1}^{n} dx_i \wedge dy_i\),

2. the metric \(ds^2\) satisfies \(|ds^2(q) - ds_0^2| \leq C|q|^2\), where \(ds_0^2 = dx_i^2 + dy_i^2\),

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3. the complex structure $J$ satisfies $|J(q) - J_0| \leq C|q|^2$, where $J_0$ is the standard complex structure in $\mathbb{R}^{2n}$,

4. the parallel holomorphic $(n,0)$ form $\Omega$ satisfies $|\Omega(q) - dZ| \leq C|q|^2$,
where $dZ = dz_1 \wedge \cdots \wedge dz_n$ and $z_j = x_j + iy_j$, $j = 1, \cdots, n$.

Assume that the two tangent planes at $p$ are $P_1$ and $P_2$ respectively. Then $P_1$ and $P_2$ are special Lagrangian and $L \cap B_{r_0}$ is Lagrangian with respect to the standard symplectic structure in $\mathbb{R}^{2n}$. It follows that $L \cap B_{r_1}$ can be written as the graph of the gradient of a function

$$\psi : P_i \cap B_{r_1} \to \mathbb{R}, \quad i = 1, 2,$$

for some $r_1 < r_0$. Moreover, we can choose $\psi$ satisfying

$$|\psi(x)| \leq K|x|^3, \quad |\nabla \psi(x)| \leq K|x|^2, \quad |\nabla^2 \psi(x)| \leq K|x|, \quad |\nabla^3 \psi(x)| \leq K,$$

and $|\nabla^4 \psi(x)| \leq C_K$ for $x \in P_i$ with $|x| \leq r_1$, where $K$ is a constant depending on the curvature of $L$ in $B_{r_1}$ and $C_K$ depends on the derivative of $K$. There exists a Lawlor neck in $\mathbb{R}^{2n}$ with suitable $a = (a_1, \cdots, a_n)$ that is asymptotic to $P_1$ and $P_2$, when $P_1 = P_0$, $-P_2 = P_{-\theta}$, and $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$.

By the discussion in last section, this condition is always satisfied when $n = 2$ or $3$. From now on, we focus on the situations in $n \geq 2$ where we can find a Lawlor neck to approximate the pair of tangent planes. We first scale the Lawlor neck $M_\alpha$ by $\varepsilon$. Outside a small ball $B_{\varepsilon R_0}$, the manifold $\varepsilon M_\alpha$ can be written as the graph of the gradient of $\Psi_\varepsilon$ over $P_1$ and $P_2$. To match $\psi$ and $\Psi_\varepsilon$ together, A. Butscher has the following estimate:

**Lemma 1** [2] There exist constants $a_0$ and $c$ depending on $L$ only, such that if $0 < \alpha < a_0$, $\tau = \frac{a}{R}$, and $\varepsilon < c \alpha^{1+\frac{1}{n}}$, then $|\nabla^2 \psi(x)| \leq \alpha$ and $|\nabla^2 \Psi_\varepsilon(x)| \leq \alpha$ for any $x \in P_i$ with $\frac{\tau}{2} \leq |x| \leq \tau$.
Roughly speaking, we want the approximate submanifolds to be $\epsilon M_a$ near $p$, and to be $L$ outside a neighborhood of $p$. We also want to require the interpolation to be Lagrangian. Recall that the graph of the gradient of a function on a Lagrangian plane is always Lagrangian. Hence the following combination of $\psi$ and $\Psi_x$ is a good candidate for our purpose. First, assume that $\eta$ is a smooth function on $\mathbb{R}^n$ satisfying $\eta(x) \equiv 1$ when $|x| \leq \frac{r}{2}$ and $\eta(x) \equiv 0$ when $|x| \geq \frac{3r}{4}$. Moreover, it also satisfies

$$0 \leq \eta(x) \leq 1, \quad |\nabla \eta(x)| \leq \frac{C}{r}, \quad |\nabla^2 \eta(x)| \leq \frac{C}{r^2}, \quad |\nabla^3 \eta(x)| \leq \frac{C}{r^3},$$

and $|\nabla^4 \eta(x)| \leq \frac{C}{r^4}$ for every $x$. Next define the interpolation to be the graph

$$T_i = \{(x, \nabla[(1 - \eta)\psi + \eta\Psi_x](x)) \in P_i \times P_i^\perp, \quad \frac{r}{2} \leq |x| \leq r\}, \quad i = 1, 2.$$ 

It is easy to check that

$$|\nabla[(1 - \eta)\psi + \eta\Psi_x]| < Cr^2, \quad \text{for} \quad \frac{r}{2} \leq |x| \leq r.$$ 

Denote

$$B'_r = B^{\mathbb{R}^n}_r \times \mathbb{R}^n \cap B^{\mathbb{R}^n}_r \times \mathbb{R}^n \subset P_1 \times P_1^\perp \cap P_2 \times P_2^\perp,$$

where $B^{\mathbb{R}^n}_r = B_r \cap P_i, \quad i = 1, 2$. We then define the approximate submanifold to be

$$M_a = (\epsilon M_a \cap B'_r) \cup T_1 \cup T_2 \cup (L \setminus B'_r).$$

The approximate submanifold is Lagrangian and satisfies the following properties:

$$\begin{cases} |H(q)| \leq \epsilon |q| & \text{for} \quad q \in \epsilon M_a \cap B'_r, \\ |H(q)| \leq C & \text{for} \quad q \in T_1 \cup T_2, \\ |H(q)| = 0 & \text{for} \quad q \in L \setminus B'_r. \end{cases}$$
where $H$ is the mean curvature vector of $M_\alpha$ in $N$. One also has

\[
\begin{align*}
|\text{Im } \Omega|_{M_\alpha}(q)| & \leq C|q|^2 \quad \text{for } q \in \partial M_\alpha \cap B_{\frac{3}{2}} \\
|\text{Im } \Omega|_{M_\alpha}(q)| & \leq C\alpha \quad \text{for } q \in T_1 \cup T_2 \\
|\text{Im } \Omega|_{M_\alpha}(q)| & = 0 \quad \text{for } q \in L \setminus B_r
\end{align*}
\]

The situation in $\mathbb{R}^{2n}$ is computed in [2]. Because $|\Omega(q) - dZ| \leq C|q|^2$ and $|H(q) - H_0(q)| \leq C|q|$, where $H_0$ is the mean curvature vector of $M_\alpha \cap B_r$ in $\mathbb{R}^{2n}$ with the Euclidean metric, we thus obtain the above estimates.

We now investigate some properties of the approximate submanifolds $M_\alpha$. From the construction, it is easy to see that they are embedded Lagrangian submanifolds with area uniformly bounded from above and below. Moreover, because $M_\alpha$ converges to $L$ in Hausdorff distance, we have $\int_{M_\alpha} \text{Im } \Omega = 0$. For a submanifold $M^n \subset \mathbb{R}^4$, J. H. Michael and L. Simon [10] proved the Sobolev inequality:

\[
\left( \int_M h^{\frac{n-1}{n-4}} dV \right)^{\frac{n-4}{n-1}} \leq C(n) \int_M \left( |\nabla^M h| + h|\bar{H}| \right) dV,
\]

where $h > 0$ is a $C^1$ function on $M$ with compact support and $\bar{H}$ is the mean curvature of $M$ in $\mathbb{R}^4$. By embedding $N$ isometrically in $\mathbb{R}^4$, the corresponding mean curvature $H_\alpha$ of $M_\alpha$ in $\mathbb{R}^4$ is uniformly bounded. Thus the Sobolev constant on $M_\alpha$ is uniformly bounded. (See the discussion in the Appendix.) We then can prove the following estimate concerning the first eigenvalue.

**Theorem 1** When $\alpha$ is small enough, the first eigenvalue $\lambda_1(M_\alpha)$ for the Laplace operator on $M_\alpha$ is bounded below by $\frac{1}{4}\lambda_1(L)$. 

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Proof. Suppose that $f_\alpha$ is the first eigenfunction for the Laplace operator on $M_\alpha$ satisfying

$$\int_{M_\alpha} f_\alpha \, dV = 0, \quad \int_{M_\alpha} f_\alpha^2 \, dV = 1, \quad \text{and} \quad \int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2 \, dV = \lambda_1(M_\alpha).$$

Because $\Delta_{M_\alpha} f_\alpha = -\lambda_1(M_\alpha) f_\alpha$, one has

$$\Delta_{M_\alpha} f_\alpha^2 = -2\lambda_1(M_\alpha) f_\alpha^2 + |\nabla^{M_\alpha} f_\alpha|^2 \geq -2\lambda_1(M_\alpha) f_\alpha^2.$$

Assume that the theorem is not true. Then there exists a subsequence $\{\alpha_j\}$ which tends to zero, such that $\lambda_1(M_{\alpha_j}) < \frac{1}{4}\lambda_1(L)$. By Lemma 5 in the Appendix, it follows that

$$f_{\alpha_j}^2 \leq C \int_{M_\alpha} f_{\alpha_j}^2 \, dV \leq C.$$

Since $\lambda_1(M_{\alpha_j})$ and $Vol(M_{\alpha_j})$ are bounded uniformly, the constant $C$ is independent of $j$.

When $n > 2$, let $\varphi_\delta$ be a nonnegative function in $N$ satisfying $\varphi_\delta \equiv 1$ on $N \setminus B_\delta$, $\varphi_\delta \equiv 0$ on $B_\frac{\delta}{2}$, $0 \leq \varphi_\delta \leq 1$ on $B_\delta \setminus B_\frac{\delta}{2}$, and $|\nabla^N \varphi_\delta| \leq \frac{3}{\delta}$. A direct computation shows that

$$\int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}|^2 \, dV$$

$$= \int_{M_{\alpha_j}} (|\nabla^{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}|^2 + \varphi_\delta^2 |\nabla^{M_{\alpha_j}} f_{\alpha_j}|^2 + 2\varphi_\delta f_{\alpha_j} \nabla^{M_{\alpha_j}} \varphi_\delta \cdot \nabla^{M_{\alpha_j}} f_{\alpha_j}) \, dV$$

$$\leq 2 \int_{M_{\alpha_j}} \varphi_\delta^2 |\nabla^{M_{\alpha_j}} f_{\alpha_j}|^2 \, dV + 2 \int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} \varphi_\delta|^2 f_{\alpha_j}^2 \, dV$$

$$\leq 2 \int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} f_{\alpha_j}|^2 \, dV + 2 \int_{M_{\alpha_j} \cap B_\delta \setminus B_\frac{\delta}{2}} |\nabla^{M_{\alpha_j}} \varphi_\delta|^2 f_{\alpha_j}^2 \, dV$$

$$\leq 2\lambda_1(M_{\alpha_j}) + \frac{C}{\delta^2} Vol(M_{\alpha_j} \cap B_\delta \setminus B_\frac{\delta}{2})$$

$$\leq 2\lambda_1(M_{\alpha_j}) + C\delta^{n-2}.$$
In the above estimates, we use $|\nabla^{M_{\alpha_j}} \varphi_\delta| \leq |\nabla^{N} \varphi_\delta|$ and $\text{Vol}(M_{\alpha_j} \cap B_\delta) \leq C \delta^n$ by monotonicity formula [14]. We also have

$$\int_{M_{\alpha_j}} (\varphi_\delta f_{\alpha_j})^2 \, dV \geq 1 - \int_{M_{\alpha_j} \cap B_\delta} f_{\alpha_j}^2 \, dV \geq 1 - C \text{Vol}(M_{\alpha_j} \cap B_\delta) \geq 1 - C \delta^n,$$

and

$$\left( \int_{M_{\alpha_j}} \varphi_\delta f_{\alpha_j} \, dV \right)^2 = \left( \int_{M_{\alpha_j} \cap B_\delta} (1 - \varphi_\delta) f_{\alpha_j} \, dV \right)^2 \leq C \text{Vol}(M_{\alpha_j} \cap B_\delta) \leq C \delta^n.$$

Recall that $M_{\alpha_j}$ is the same as $L$ in $N \setminus B_{\frac{1}{2}}$ for $\alpha_j \leq \frac{K \delta}{2}$. Therefore,

$$\frac{\int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}|^2 \, dV}{\int_{M_{\alpha_j}} (\varphi_\delta f_{\alpha_j})^2 \, dV - (\int_{M_{\alpha_j}} \varphi_\delta f_{\alpha_j} \, dV)^2} \geq \frac{\int_{L} |\nabla^{L} \varphi_\delta f_{\alpha_j}|^2 \, dV}{\int_{L} (\varphi_\delta f_{\alpha_j})^2 \, dV - (\int_{L} \varphi_\delta f_{\alpha_j} \, dV)^2} \geq \lambda_1(L).$$

On the other hand, it follows from the above estimates that

$$\frac{\int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}|^2 \, dV}{\int_{M_{\alpha_j}} (\varphi_\delta f_{\alpha_j})^2 \, dV - (\int_{M_{\alpha_j}} \varphi_\delta f_{\alpha_j} \, dV)^2} \leq \frac{2 \lambda_1(M_{\alpha_j}) + C \delta^{n-2}}{1 - C \delta^n}.$$

Choose $\delta$ small enough so that $C \delta^{n-2} < \min(\frac{\lambda_1(L)}{4}, \frac{1}{4})$. Then by combining the two inequalities, one gets $\lambda_1(M_{\alpha_j}) > \frac{1}{4} \lambda_1(L)$ when $\alpha_j \leq \frac{K \delta}{2}$, which is a contradiction. Thus the theorem is proved in the case $n > 2$.

When $n = 2$, we need to modify the function $\varphi_\delta$ as follows:

$$\varphi_\delta(x) = \begin{cases} 
0 & |x| < \delta^2 \\
\frac{\log \frac{|x|}{\delta^2}}{\log \frac{1}{\delta}} & \delta^2 \leq |x| \leq \delta \\
1 & |x| > \delta \end{cases}$$

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A direct computation gives

$$\int_{M_{\alpha_j} \cap B_\delta \setminus B_{\delta^2}} |\nabla^{M_{\alpha_j}} \varphi_\delta|^2 dV \leq \frac{C}{|\log \delta|}.$$ 

Recall that $M_{\alpha_j}$ is the same as $L$ in $N \setminus B_{\delta^2}$ for $\alpha_j \leq K\delta^2$. Similar arguments as in the case $n > 2$ lead to

$$\lambda_1(L) \leq \frac{2\lambda_1(M_{\alpha_j}) + \frac{C}{|\log \delta|}}{1 - C\delta^n}.$$ 

Choose $\delta$ small enough so that $\frac{C}{|\log \delta|} < \frac{\lambda_1(L)}{4}$ and $C\delta^n \leq \frac{1}{4}$. One will get

$$\lambda_1(M_{\alpha_j}) > \frac{1}{4} \lambda_1(L)$$

when $\alpha_j \leq K\delta^2$, which is a contradiction. This completes the proof of the theorem.

Q.E.D.

Remark: It is easy to see from the proof that the lower bound can be improved and the estimate also works for other singularities. Because the submanifold $L$ is compact and connected, its first eigenvalue $\lambda_1(L)$ is a positive number.

4 Perturbation

There exists a constant $c_1$ such that the exponential map from the normal bundle $T_{c_1}^\perp M_{\alpha_j}$ into $N$ is an embedding in the $c_1 \varepsilon$ neighborhood of $M_{\alpha_j}$. Choose a smooth function $\eta_\alpha$ such that $\eta_\alpha(s) \equiv 1$ when $|s| \leq \frac{c_1 \varepsilon}{2}$, and $\eta_\alpha(s) \equiv 0$ when $|s| \geq \frac{3c_1 \varepsilon}{4}$. Moreover, it also satisfies

$$0 \leq \eta_\alpha(s) \leq 1, \quad |\nabla \eta_\alpha(s)| \leq \frac{C}{\varepsilon}, \quad |\nabla^2 \eta_\alpha(s)| \leq \frac{C}{\varepsilon^2}, \quad \text{and} \quad |\nabla^3 \eta_\alpha(s)| \leq \frac{C}{\varepsilon^3}.$$
for every s. Given a $C^{2,\beta}$ function $u$ on $M_\alpha$, $0 \leq \beta \leq 1$, we can extend it into a $C^{2,\beta}$ function $U$ on $N$ by defining $U(\exp(x,v)) = \eta(x(|v|)u(x)$ for $x \in M_\alpha$ and $v \in T_x^1M_\alpha$. We then solve the Hamiltonian flow:

$$\frac{\partial \phi(t,q)}{\partial t} = J\nabla^NU(\phi(t,q)) \quad \text{and} \quad \phi(0,q) = q \quad \text{for} \quad q \in N.$$ 

There exists a unique solution $C^{1,\beta}$ for small $t$. Note that if $\phi_U(t,q)$ is a solution defined by $U$, then $\phi_U(st,q)$ is a solution defined by $sU$. Denote

$$\phi_u(x) = \phi_U(1,x) \quad \text{for} \quad x \in M_\alpha.$$ 

The map $\phi_u$ can be defined for $u$ in a neighborhood of the zero function. In particular, it is defined when $\|(\nabla^N)^2U\|_{0,N} < 1$. Because $\phi_U(1,q)$ is a symplectic map, the image $\phi_u(M_\alpha)$ is Lagrangian. Moreover, the family of maps $\phi_\tau$, $0 \leq t \leq 1$, is a homotopy between $\phi_u$ and $\phi_0$. Define a $C^{0,\beta}$ function on $M_\alpha$ by $F_u(x) = *\phi_u^*(\text{Im } \Omega)(x)$, where $*$ is the star operator with respect to the induced metric on $M_\alpha$. If we can find a function $u$ such that $\phi_u$ is an embedding and satisfies $F_u(u) = 0$, then $\phi_u(M_\alpha)$ will be an embedded special Lagrangian submanifold. Therefore, the goal is to find the zero set of $F_u$. The differential of $F_u$ at the zero function is

$$DF_u(0)(u) = *\phi_0^*(d\,i_{\nabla^N}U\text{Im } \Omega)(x).$$

Because $M_\alpha$ is Lagrangian, there exists a function $\theta(x)$ (mod $2\pi$) on $M_\alpha$, such that

$$\Omega|_{M_\alpha} = e^{i\theta(x)} \omega_1 \wedge \cdots \wedge \omega_n.$$
where \( \omega_1 \cdots \omega_n \) is a local orthonormal basis on the cotangent bundle \( T^* M_\alpha \) [13]. Note that

\[
\phi_0^i(i_{J\nabla U} \text{Im } \Omega) = \text{Im } \sum_{\beta=1}^n e^{i\theta(x)} \left[ i (J\nabla U)^n + \omega_\beta \wedge \cdots \wedge \omega_\beta \cdots \wedge \omega_n \right] \\
+ (J\nabla U)^n \omega_1 \wedge \cdots \wedge \omega_\beta \cdots \wedge \omega_n
\]

\[= \cos \theta(x) \star du,
\]

where \( \omega_\beta \) means that \( \omega_\beta \) does not appear and the last equality follows from the definition of \( U \). Because \( H = J\nabla M_\alpha \theta \) [13], we thus have

\[
DF_{\alpha}(0)(u) = \cos \theta(x) \Delta M_\alpha u - \sin \theta(x) < H, J\nabla M_\alpha u >.
\]

It will be denoted by \( Lu \) for simplicity. Because \( |\sin \theta| = |\phi_0^i(\text{Im } \Omega)| \leq C \alpha \), it follows that \( |\phi(x)| \leq C \alpha \). One then can show

**Proposition 2** When \( \alpha \) is small, the operator \( L \) is an elliptic operator and its kernel consists of the constant functions. Moreover, the first eigenvalue \( \lambda_1(M_\alpha, L) \) for the operator \( L \) on \( M_\alpha \) has a uniform positive lower bound.

**Proof.** When \( \alpha \) is small, \( \cos \theta(x) \) is close to 1 and hence \( L \) is an elliptic operator. Constants are clearly in the kernel of \( L \). Suppose that \( Lu = 0 \) and \( \int_{M_\alpha} u dV = 0 \) (i.e., normalize \( u \) such that it is perpendicular to constants). Because \( \cos \theta(x) \) is nonzero, \( Lu = 0 \) is equivalent to

\[
\Delta M_\alpha u - \tan \theta(x) < H, J\nabla M_\alpha u > = 0.
\]
Multiply $u$ on both sides, and integrate over $M_\alpha$. We then get
\[
\int_{M_\alpha} |\nabla^{M_\alpha} u|^2 \, dV = - \int_{M_\alpha} u \Delta_{M_\alpha} u \, dV
\]
\[
= - \int_{M_\alpha} u \tan \theta(x) < H, J \nabla^{M_\alpha} u > \, dV
\]
\[
\leq C \max_{M_\alpha} (\tan \theta(x)) \int_{M_\alpha} |u| |\nabla^{M_\alpha} u| \, dV
\]
\[
\leq C \max_{M_\alpha} (\tan \theta(x)) (\int_{M_\alpha} |u|^2 \, dV)^{1/2} (\int_{M_\alpha} |\nabla^{M_\alpha} u|^2 \, dV)^{1/2}.
\]
We use the fact that $|H|$ is bounded in the first inequality above. When $\alpha$ tends to zero, the number $\max_{M_\alpha} (\tan \theta(x))$ also tends to zero. On the other hand, the first eigenvalue $\lambda_1(M_\alpha)$ for the Laplace operator on $M_\alpha$ is bounded below by $\frac{1}{4} \lambda_1(L)$ from Theorem 1. It implies that $u$ is identically zero when $\alpha$ is sufficiently small. Thus the kernel of $\mathcal{L}$ consists of only constant solutions.

We now estimate $\lambda_1(M_\alpha, \mathcal{L})$. Suppose that $f_\alpha$ is the first eigenfunction of $\mathcal{L}$, which satisfies
\[
\int_{M_\alpha} f_\alpha \, dV = 0, \quad \int_{M_\alpha} f_\alpha^2 \, dV = 1, \quad \text{and} \quad \mathcal{L} f_\alpha = -\lambda_1(M_\alpha, \mathcal{L}) f_\alpha.
\]
By choosing $\alpha$ small enough, we can assume that $\cos \theta(x) > \frac{1}{2}$. Multiply both sides of the equation by $-\frac{f_\alpha}{\cos \theta(x)}$ and integrate over $M_\alpha$. We have
\[
- \int_{M_\alpha} f_\alpha \Delta_{M_\alpha} f_\alpha \, dV + \int_{M_\alpha} f_\alpha \tan \theta(x) < H, J \nabla^{M_\alpha} f_\alpha > \, dV
\]
\[
= \lambda_1(M_\alpha, \mathcal{L}) \int_{M_\alpha} \frac{f_\alpha^2}{\cos \theta(x)} \, dV.
\]
A direct computation shows that
\[
\left| \int_{M_\alpha} f_\alpha \tan \theta(x) < H, J \nabla^{M_\alpha} f_\alpha > \, dV \right|
\]
\[
\leq C \max_{M_\alpha} (\tan \theta(x)) (\int_{M_\alpha} |f_\alpha|^2 \, dV)^{1/2} (\int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2 \, dV)^{1/2},
\]
\[
\leq C \max_{M_\alpha} (\tan \theta(x)) (\int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2 \, dV)^{1/2}.
\]
Plugging this into the above equality, we will get

\[ 2\lambda_1(M_\alpha, \mathcal{L}) \geq \lambda_1(M_\alpha, \mathcal{L}) \int_{M_\alpha} \frac{f_\alpha^2}{\cos \theta(x)} \, dV \]

\[ \geq \int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2 \, dV - C \max_{M_\alpha} (\tan \theta(x))(\int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2 \, dV)^{\frac{1}{2}} \]

\[ \geq \frac{1}{2} \int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2 \, dV \]

\[ \geq \frac{1}{2} \lambda_1(M_\alpha), \]

when \( \alpha \) is sufficiently small. This completes the proof of the proposition.

Q.E.D.

5 The theorem

We first set some notation which will be used in the rest of this paper. Assume that \( u \) is a function on \( M_\alpha \). We denote

\[ \|u\|_{0, M_\alpha} = \sup_{M_\alpha} |u|, \]

\[ |u|_{\beta, M_\alpha} = \sup_{x, x' \in M_\alpha} \frac{|u(x) - u(x')|}{\text{dist}(x, x')^\beta}, \quad 0 < \beta < 1, \]

and

\[ \|u\|_{L^p} = \left( \int_{M_\alpha} u^p \, dV \right)^{\frac{1}{p}}. \]

We can embed \( N \) isometrically into \( R^t \) and set

\[ |(\nabla^{M_\alpha})^k u|_{\beta, M_\alpha} = \sup_{x, x' \in M_\alpha} \frac{|(\nabla^{M_\alpha})^k u(x) - (\nabla^{M_\alpha})^k u(x')|}{\text{dist}(x, x')^\beta}, \]

where \( k \) is a positive integer.
When \( \alpha \) tends to zero, the neck on \( M_\alpha \) will shrink to \( p \). Thus we need to introduce a weighted norm to do the estimates. Roughly speaking, we want to choose the weight function \( \rho(x) \) on \( M_\alpha \) such that \( \rho(x) \) is less than the radius of a normal ball at \( x \in M_\alpha \). More precisely, we can choose that \( \rho(x) \) is of the form [2]:

\[
\rho(x) = \begin{cases} 
  c \varepsilon & \text{for } x \in M_\alpha \cap B_{r_2} \\
  \text{interpolation} & \text{for } x \in M_\alpha \cap B_{r_2} \setminus B_{r_2} \\
  R_2 & \text{for } x \in M_\alpha \setminus B_{r_2}
\end{cases}
\]

for some constants \( r_2 \) and \( R_2 \). In addition, we can also require \( \rho(x) \) to satisfy the following properties:

1. \( \| \nabla^{M_\alpha} \rho \|_{0,M_\alpha} \leq C \),

2. \( c \alpha \leq \rho(x) \leq C \alpha \) for \( x \in T_1 \cup T_2 \),

3. \( \| \rho^{-1} \|_{L^p} \leq C \) for \( p < n \).

**Definition 3** Let \( u \) be a \( C^{k,\beta} \) function on \( M_\alpha \), where \( k \) is an integer and \( 0 < \beta < 1 \). The \( \rho \)-weighted \( (k,\beta) \) norm \( \| u \|_{C^{k,\beta}(M_\alpha)} \) of \( u \) is defined as the sum:

\[
\| u \|_{0,M_\alpha} + \| \rho |\nabla^{M_\alpha} u| \|_{0,M_\alpha} + \cdots + \| \rho^k |(\nabla^{M_\alpha})^k u| \|_{0,M_\alpha} + \| \rho^{k+\beta} (\nabla^{M_\alpha})^k u \|_{\beta,M_\alpha}.
\]

**Proposition 3** The operator \( \mathcal{L} \) is a bounded operator between the Banach space \( C^{2,\beta}(M_\alpha) \) with norm \( \| \cdot \|_{C^{2,\beta}(M_\alpha)} \) and the Banach space \( C^{0,\beta}(M_\alpha) \) with norm \( \| \cdot \|_{C^{0,\beta}(M_\alpha)} \).

**Proof.** Note that

\[
\| \rho^2 \mathcal{L} u \|_{C^{0,\beta}(M_\alpha)} \leq \| \rho^2 \cos \theta \Delta_{M_\alpha} u \|_{C^{0,\beta}(M_\alpha)} + \| \rho^2 \sin \theta < H, J \nabla^{M_\alpha} u \|_{C^{0,\beta}(M_\alpha)}.
\]
A direct computation gives
\[
\|\rho^2 \cos \theta \Delta_M u\|_{C^0_p(M)} \\
\leq \|\rho^2 \Delta_M u\|_{0,M} + [\rho^{2-\beta} \Delta_M u]_{\beta,M} + [\cos \theta]_{\beta,M} \|\rho^2\|_{0,M} \|\rho^2 \Delta_M u\|_{0,M} \\
\leq C \|u\|_{C^2_p(M)}.
\]
We also have
\[
\|\rho^2 \sin \theta < H, J \nabla^M u > \|_{C^0_p(M)} \\
\leq \|\rho \sin \theta |H| \|_{0,M} \|\nabla^M u\|_{0,M} + \|\rho \sin \theta |H| \|_{0,M} [\rho^{1-\beta} \nabla^M u]_{\beta,M} \\
+ [\rho \sin \theta H]_{\beta,M} \|\rho^2\|_{0,M} \|\rho \|_{0,M} \|\nabla^M u\|_{0,M}.
\]
Using the fact that the mean curvature is zero outside a small ball and the properties of \(\rho\) and \(\sin \theta\), it follows that
\[
\|\rho^2 \sin \theta < H, J \nabla^M u > \|_{0,M} \leq C \alpha^2 \|\nabla^M u\|_{0,M}.
\]
Moreover, when we estimate \(\|\rho^2 \sin \theta < H, J \nabla^M u > \|_{C^0_p(M)}\), all the suprema involved can be taken only over the small ball. When \(\text{dist}(x, x') \geq \alpha\), one has that
\[
\frac{|\rho \sin \theta H(x) - \rho \sin \theta H(x')|}{\text{dist}(x, x')^{\beta}} \leq C \alpha^{2-\beta}.
\]
Note that \(\|\nabla^M u\|^2 \sin \theta\|_{0,M} \leq C \alpha^{-1}\). Hence when \(\text{dist}(x, x') \leq \alpha\), one has
\[
\frac{|\rho \sin \theta H(x) - \rho \sin \theta H(x')|}{\text{dist}(x, x')^{\beta}} \leq C \alpha \alpha^{1-\beta} = C \alpha^{2-\beta}.
\]
Therefore,
\[
\|\rho^2 \sin \theta < H, J \nabla^M u > \|_{C^0_p(M)} \leq C \alpha^2 \|u\|_{C^1_p(M)}.
\]
We thus have
\[
\|\rho^2 \mathcal{L} u\|_{C^0_p(M)} \leq C \|u\|_{C^2_p(M)}.
\]
By the elliptic Schauder estimate \cite{2} for the $\rho$-weighted $(k, \beta)$ norms, one can prove that
\[
\|u\|_{C^2_{p,\beta}(M_\alpha)} \leq C\varepsilon^{-\beta}(\|\rho^2 \Delta_{M_\alpha} u\|_{C^0_{p,\beta}(M_\alpha)} + \|u\|_{0,M_\alpha}).
\]

In the Appendix, we show that $\|u\|_{0,M_\alpha} \leq C\varepsilon^{-\nu}\|\rho^2 \Delta_{M_\alpha} u\|_{C^0_{p,\beta}(M_\alpha)}$ for $u$ satisfying $\int_{M_\alpha} u \, dV = 0$. We thus have
\[
\|u\|_{C^2_{p,\beta}(M_\alpha)} \leq C\varepsilon^{-[\beta+\nu]}\|\rho^2 \Delta_{M_\alpha} u\|_{C^0_{p,\beta}(M_\alpha)}.
\]

In the next Lemma, we bound $\|\rho^2 \Delta_{M_\alpha} u\|_{C^0_{p,\beta}(M_\alpha)}$ by $\|\rho^2 L u\|_{C^0_{p,\beta}(M_\alpha)}$ and hence obtain

**Lemma 2** Suppose that $u$ is a $C^{2,\beta}$ function on $M_\alpha$, $0 < \beta < 1$, which satisfies $\int_{M_\alpha} u \, dV = 0$. Then when $\alpha$ is small, one has that
\[
\|u\|_{C^2_{p,\beta}(M_\alpha)} \leq C\varepsilon^{-[\beta+\nu]}\|\rho^2 L u\|_{C^0_{p,\beta}(M_\alpha)},
\]
where $\nu$ is any positive number. The constant $C$ depends on $\nu$, but is independent of $\alpha$.

**Proof.** Note that
\[
\|\rho^2 L u\|_{C^0_{p,\beta}(M_\alpha)} \geq \|\rho^2 \cos \theta \Delta_{M_\alpha} u\|_{C^0_{p,\beta}(M_\alpha)} - \|\rho^2 \sin \theta < H, J \nabla M_\alpha u \|_{C^0_{p,\beta}(M_\alpha)},
\]
and
\[
\|\rho^2 \cos \theta \Delta_{M_\alpha} u\|_{C^0_{p,\beta}(M_\alpha)} = \|\rho^2 \cos \theta \Delta_{M_\alpha} u\|_{0,M_\alpha} + [\rho^{2+\beta} \cos \theta \Delta_{M_\alpha} u]_{\beta,M_\alpha}.
\]

When $\alpha$ is small, we have
\[
\|\rho^2 \cos \theta \Delta_{M_\alpha} u\|_{0,M_\alpha} \geq \frac{1}{2} \|\rho^2 \Delta_{M_\alpha} u\|_{0,M_\alpha}.
\]
and

\[ [\rho^{2+\beta} \cos \theta \Delta_{M_0} u]_{\beta,M_0} \geq \frac{1}{2} [\rho^{2+\beta} \Delta_{M_0} u]_{\beta,M_0} - C_{\alpha} \| \rho^2 \Delta_{M_0} u \|_{0,M_0}. \]

Hence

\[ \| \rho^2 \cos \theta \Delta_{M_0} u \|_{C^{2,\beta}_2(M_0)} \geq \frac{1}{3} \| \rho^2 \Delta_{M_0} u \|_{C^{2,\beta}_2(M_0)}. \]

On the other hand, we have

\[ \| \rho^2 \sin \theta < H, J \nabla u \|_{C^{2,\beta}_2(M_0)} \leq C_{\alpha} \| u \|_{C^{1,\beta}_2(M_0)} \]

\[ \leq C_{\alpha} \varepsilon^{-\nu} \| \rho^2 \Delta_{M_0} u \|_{C^{0,\beta}_2(M_0)}. \]

Putting all these estimates together, we get

\[ \| \rho^2 \mathcal{L} u \|_{C^{2,\beta}_2(M_0)} \geq \frac{1}{4} \| \rho^2 \Delta_{M_0} u \|_{C^{2,\beta}_2(M_0)} \]

and the proposition is therefore proved.

Q.E.D.

**Remark:** Note that in the proof we first need to fix \( \nu \) and then choose \( \alpha \) small enough.

Denote the Banach space of \( C^{2,\beta} \) functions on \( M_0 \) which satisfies \( \int_{M_0} u \, dV = 0 \) with norm \( \| \cdot \|_{C^{2,\beta}_2(M_0)} \) by \( B_1 \) and the Banach space of \( C^{0,\beta} \) functions on \( M_0 \) which satisfies \( \int_{M_0} u \, dV = 0 \) with norm \( \| \rho^2 \cdot \|_{C^{0,\beta}_2(M_0)} \) by \( B_2 \). Because \( \int_{M_0} \text{Im} \Omega = 0 \) and the family of maps \( \phi_t \), \( 0 \leq t \leq 1 \), is a homotopy between \( \phi_u \) and \( \phi_0 \), it follows that \( \int_{M_0} \mathcal{F}_u (u) \, dV = 0 \). Thus we can restrict \( \mathcal{F}_u \) as a map from \( B_1 \) into \( B_2 \). A direct computation shows that the operator \( \mathcal{L} \) is self-adjoint. By Proposition 2, we consequently have:

**Proposition 4** The operator \( \mathcal{L} \) from \( B_1 \) into \( B_2 \) is injective and surjective.
We will apply the following version of inverse function theorem to $F_a$.

**Theorem 2** [1] Let $F : B \to B'$ be a $C^1$ map between Banach spaces and suppose that the differential $DF(0)$ of $F$ at 0 is an isomorphism. Moreover, suppose that $F$ satisfies the estimates:

1. $\|DF(0)x\|_{B'} \geq C_L \|x\|_B$ for any $x \in B$,

2. $\|DF(0)y - DF(x)y\|_{B'} \leq C_N \|x\|_B \|y\|_B$ for all $x$ sufficiently near 0 and for any $y \in B$,

where $C_L$ and $C_N$ are constants independent of $x$ and $y$. Then there exist neighborhoods $U$ of 0 and $V$ of $F(0)$ so that $F : U \to V$ is a $C^1$ diffeomorphism and $V$ contains the ball $B_{C_F r}(F(0))$, where $r \leq \frac{C_F}{\sqrt{C_N}}$. Furthermore, the image of the ball $B_r(0)$ under $F$ contains the ball $B_{C_F r}(F(0))$.

We already get an estimate on $C_L$ in Lemma 2 and still need an estimate on $C_N$ to apply Theorem 2.

**Lemma 3** Assume that $v \in B_1$ and is sufficiently near 0. The differential of $F_a$ at $v$ satisfies the following estimate:

$$\left\|\rho^2(DF_a(v)(u) - DF_a(0)(u))\right\|_{C^{0,\beta}_p(M_a)} \leq C\varepsilon^{-\beta}\|v\|_{C^{2,\alpha}_p(M_a)}\|u\|_{C^{2,\alpha}_p(M_a)}$$

for all $u \in B_1$.

**Remark:** The lemma is also proved in [3]. But the proof and the estimate obtained are slightly different.

**Proof.** Suppose that the Riemannian metric on $N$ is $g$. Define a conformal metric $g' = s^{-2}g$, where $s$ is a constant. Assume that the Hamiltonian
flow determined by \( v \) with respective to the metric \( g \) is the same as the Hamiltonian flow determined by a function \( v_s \) with respect to the metric \( g' \), which is denoted by \( \phi'_{tv_s} \). Then

\[
J \nabla^N v = \frac{d\phi_{tv}}{dt} = \frac{d\phi'_{tv_s}}{dt} = J \nabla^{(N,g')} v_s = s \ J \nabla^N v_s,
\]

where \( \nabla^{(N,g')} \) is the covariant derivative with respect to the metric \( g' \). It implies that \( v_s = s^{-1} v \). Denote \( s^{-n} \Omega \) by \( \Omega' \), which is a unit length holomorphic \((n,0)\) form on \((N,g')\). Define \( G_\alpha(v_s) = \ast' (\phi'_{v_s})^* (\text{Im} \Omega') \), where \( \ast' \) is the star operator with respect to the metric on \( M_\alpha \) induced from \((N,g')\). Since

\[
\ast' (\phi'_{v_s})^* (\text{Im} \Omega') = \ast' s^{-n} \phi'_{v_s} (\text{Im} \Omega) \\
= \ast s^{-n} \phi_{v_s} (\text{Im} \Omega) \\
= \ast \phi_{v_s} (\text{Im} \Omega),
\]

one has \( G_\alpha(v_s) = F_\alpha(v) \). Assume that \( K_1 \geq \frac{1}{2} \) is an upper bound of \( \| \nabla^{M_\alpha} \rho \|_{0,M_\alpha} \). Choose \( x \in M_\alpha \) and let \( s = \rho(x) \). Then the function \( \rho \) satisfies \( \frac{s}{2} \leq \rho \leq \frac{3s}{2} \) in the ball \( B_{\frac{sK_1}{2}}(x) \) and the induced metric are bounded uniformly in this ball. Denote the norm with respect to the metric \( g' \) by \( \| \cdot \|_{g'} \). We have

\[
\| \rho^2 (\frac{dF_\alpha(v + tu)}{dt} |_{t=0} - \frac{dF_\alpha(tu)}{dt} |_{t=0}) \|_{0,B_{\frac{sK_1}{2}}(x)} \\
\leq \frac{9s^2}{4} \| D F_\alpha(v)(u) - D F_\alpha(0)(u) \|_{0,B_{\frac{sK_1}{2}}(x)} \\
= \frac{9s^2}{4} \| D G_\alpha(v_s)(u_s) - D G_\alpha(0)(u_s) \|_{0,B_{\frac{sK_1}{2}}(x)} \\
\leq C s^2 \| u_s \|_{C^2(B_{\frac{sK_1}{2}}(x))} \| v_s \|_{C^2(B_{\frac{sK_1}{2}}(x))},
\]

where in the last inequality we use the fact that \( M_\alpha \) with the metric induced
from \((N, g')\) has uniformly bounded geometry in \(B_{\frac{1}{2K_1}}(x)\). Because

\[
\|u_s\|_{C^3(B_{\frac{1}{2K_1}}(x))}^2 = \|s^{-1}u\|_{0, B_{\frac{1}{2K_1}}(x)}^2 + \|\nabla g's^{-1}u\|_{0, B_{\frac{1}{2K_1}}(x)}^2 + \|s^{-1}(\nabla v's^{-1}u)\|_{0, B_{\frac{1}{2K_1}}(x)}^2
\]

\[
= \|s^{-1}u\|_{0, B_{\frac{1}{2K_1}}(x)} + \|\nabla u\|_{0, B_{\frac{1}{2K_1}}(x)} + \|s\nabla^2 u\|_{0, B_{\frac{1}{2K_1}}(x)}
\]

\[
\leq C \|s^{-1}u\|_{c_2^2(M_\alpha)}
\]

it follows that

\[
|\rho^2(DF_\alpha(H)(u) - DF_\alpha(0)(u))\|_{0, B_{\frac{1}{2K_1}}(x)} \leq C \|H\|_{c_2^2(M_\alpha)} \|u\|_{c_2^2(M_\alpha)}
\]

We next need to estimate the following quantity \((A)\):

\[
|\rho^{2+\beta}(DF_\alpha(H)(u) - DF_\alpha(0)(u))(x) - \rho^{2+\beta}(DF_\alpha(H)(u) - DF_\alpha(0)(u))(x')|_{\text{dist}(x, x')^\beta}
\]

When \(\text{dist}(x, x') \leq \frac{1}{4K_1}\), we have

\[
(A) \leq C \|s^{-1}u\|_{c_2^2(B_{\frac{1}{2K_1}}(x))} \|Hs\|_{c_2^2(B_{\frac{1}{2K_1}}(x))}. 
\]

Since

\[
\|u_s\|_{c_2^2(B_{\frac{1}{2K_1}}(x))} \leq s^{-1}(\|u\|_{0, B_{\frac{1}{2K_1}}(x)} + \|\nabla u\|_{0, B_{\frac{1}{2K_1}}(x)} + \|s\nabla^2 u\|_{0, B_{\frac{1}{2K_1}}(x)} + s^2\|\nabla^2 u\|_{0, B_{\frac{1}{2K_1}}(x)} + s^{2+\beta}(\nabla^2 u)_{B_{\frac{1}{2K_1}}(x)})
\]

\[
\leq C \|s^{-1}u\|_{c_2^2(B_{\frac{1}{2K_1}}(x))} 
\]
and

\[
\frac{s^{2+\beta} |\nabla^2 u(x) - \nabla^2 u(x')|}{\text{dist}(x, x')^{\beta}}
\]

\leq C \frac{\rho^{2+\beta}(x) \nabla^2 u(x) - \rho^{2+\beta}(x') \nabla^2 u(x')}{\text{dist}(x, x')^{\beta}}
\]

\leq C \frac{\rho^{2+\beta}(x) \nabla^2 u(x) - \rho^{2+\beta}(x') \nabla^2 u(x') + \rho^{2+\beta}(x') \nabla^2 u(x') - \rho^{2+\beta}(x) \nabla^2 u(x')}{\text{dist}(x, x')^{\beta}}
\]

\leq C ([\rho^{2+\beta} \nabla^2 u]_{\beta, B_{\frac{R}{2}}(z)} + s^2 \|\nabla^2 u\|_{0, \tilde{B}_{\frac{R}{2}}(z)}),
\]

we thus have

\[\|u_s\|_{C^{2,\beta}(B_{\frac{R}{2}}(z))} \leq C s^{-1} \|u\|_{C^{2,\beta}(M_a)}.
\]

When \text{dist}(x, x') \geq \frac{R}{2K_1}, \text{ it is easy to see that}

\[(A) \leq C s^{-\beta} \|H\|_{C^2(M_a)} \|u\|_{C^2(M_a)}.
\]

Putting the estimates together, we therefore get

\[\|\rho^{2+\beta} (DF_a(H)(u) - DF_a(0)(u))\|_{\beta, M_a} \leq C \varepsilon^{-\beta} \|H\|_{C^2(M_a)} \|u\|_{C^{2,\beta}(M_a)}.
\]

Hence it follows that

\[\|\rho^{2} (DF_a(H)(u) - DF_a(0)(u))\|_{C^{2,\beta}(M_a)} \leq C \varepsilon^{-\beta} \|H\|_{C^2(M_a)} \|u\|_{C^{2,\beta}(M_a)}.
\]

Q.E.D.

We can choose \(\nu = \beta\) in Lemma 6. Then choose \(C_L = \frac{1}{C} \varepsilon^{2+\beta}\) by Lemma 2 and \(C_N = C \varepsilon^{-\beta}\) by Lemma 3. Applying Theorem 2, we therefore conclude that the image of the ball \(B_r(0)\) under \(F_a\) contains the ball \(B_{\frac{3\varepsilon r}{2C}}(F_a(0))\), where

\[r \leq \frac{\varepsilon^{3\beta}}{2C^2}.
\]
Lemma 4 The zero function lies in the ball \( B_{\frac{1}{2}e^{2+\alpha}(\mathcal{F}_\alpha(0))} \).

Proof. Denote \( E = \ast \phi_c(\text{Im} \Omega) = \mathcal{F}_\alpha(0) \). Recall that \( |E(x)| \leq C\alpha \) and \( E(x) = 0 \) for \( x \in M_\alpha \setminus B_{\frac{1}{2}} \). This together with the properties of \( \rho \) thus imply that

\[
\|\rho^2 E\|_{0,M_\alpha} \leq C\alpha^3.
\]

Moreover, we have

\[
|\nabla^{M_\alpha} E| = |\cos \theta \nabla^{M_\alpha} \theta| \leq |H| \leq C.
\]

Therefore, when \( \text{dist}(x, x') \leq \varepsilon \), it follows that

\[
\frac{|\rho^{2+\beta} E(x) - \rho^{2+\beta} E(x')|}{\text{dist}(x, x')^\beta} \leq C\alpha^{2+\beta} \varepsilon^{1-\beta}.
\]

When \( \text{dist}(x, x') \geq \varepsilon \), it follows that

\[
\frac{|\rho^{2+\beta} E(x) - \rho^{2+\beta} E(x')|}{\text{dist}(x, x')^\beta} \leq C\alpha^{3+\beta} \varepsilon^{-\beta}.
\]

Since \( \varepsilon = \alpha^{\frac{\beta+1}{\alpha}} \), we thus obtain

\[
\|\rho^2 E\|_{C^{2,(0)}(M_\alpha)} \leq C\alpha^{3-\beta} \leq \frac{C_L}{2} \varepsilon^{2+\beta}
\]

when \( \beta \) and \( \varepsilon \) are small enough.

Q.E.D.

The extension function \( U \) satisfies

\[
\|\nabla^\alpha U\|_{0,N} \leq C\varepsilon^{-1} (\|u\|_{0,M_\alpha} + \varepsilon \|\nabla^{M_\alpha} u\|_{0,M_\alpha}) \leq C\varepsilon^{-1} \|u\|_{C^\alpha(M_\alpha)}.
\]

\[
\|\nabla^N U\|_{0,N} \leq C\varepsilon^{-2} \|u\|_{C^\alpha(M_\alpha)}.
\]
When \( \|\nabla^N U\|_{0,N} \leq \frac{1}{2} \), or \( \|u\|_{C^2(M_\alpha)} \leq \frac{\varepsilon^2}{C} \), the map \( \phi_u \) is defined. When \( \|\nabla^N U\|_{0,N} \leq c_1 \varepsilon \), or \( \|u\|_{C^2(M_\alpha)} \leq \frac{\varepsilon^2}{C} \), then \( \phi_u(M_\alpha) \) is embedded in a \( c_1 \varepsilon \) neighborhood of \( M_\alpha \). Choose \( r = \varepsilon^{2+\beta} \leq \frac{\varepsilon^2}{2C^2} \) in Theorem 2, we knows that there exists a function \( u \in B_r \) with \( \|u\|_{C^2,1(M_\alpha)} \leq \varepsilon^{2+\beta} \) such that \( F_\alpha(u) = 0 \). It follows that \( \phi_u(M_\alpha) \) is an embedded special Lagrangian submanifold. We hence prove the main theorem of the paper:

\textbf{Theorem 3} Every compact, connected, and immersed special Lagrangian, which has only isolated transversal self-intersection points in a compact 2 or 3 dimensional Calabi-Yau manifold, is the limit of a family of embedded special Lagrangian submanifolds.

\textbf{Theorem 4} Suppose that \( L \) is a compact, connected, and immersed special Lagrangian submanifold in a compact \( n \)-dimensional Calabi-Yau manifold, \( n > 3 \). Moreover, assume that \( L \) has only isolated transversal self-intersection of two sheets and the two tangent planes at each intersection point satisfy the angle condition \( \theta_1 + \cdots + \theta_n = \frac{\pi}{2} \) (see section 2). Then \( L \) is the limit of a family of embedded special Lagrangian submanifolds.

\textbf{Appendix : Supremum Estimate}

The author would like to thank A. Butscher for informing her of the useful references [12], [15], and showing her a basic argument [4] for the De Giorgi-Nash estimates in this Appendix. We modify these arguments and present the material here for the reader's reference and completeness.

For a submanifold \( M^n \subset R^4 \), J. H. Michael and L. Simon [10] proved the
Sobolev inequality:

\[
\left( \int_M h^{\frac{n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq C(n) \int_M (|\nabla^M h| + h|\bar{H}|) \, dV,
\]

where \( h > 0 \) is a \( C^1 \) function on \( M \) with compact support and \( \bar{H} \) is the mean curvature of \( M \) in \( \mathbb{R}^n \). When \( n > 2 \), the inequality can be converted easily into

\[
\left( \int_M h^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq C(n) \left( \int_M |\nabla^M h|^2 \, dV + \int_M h^2 |\bar{H}|^2 \, dV \right).
\]

Or write as

\[
\left( \int_M h^{\frac{2n}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq C(n) \left( \int_M |\nabla^M h|^2 \, dV + \text{Vol}(M)^{-\frac{2}{n}} \int_M h^2 \, dV \right), \tag{1}
\]

where we absorb \( \sup |\bar{H}|^2 \) with \( \text{Vol}(M)^{-\frac{2}{n}} \) to make the expression scaling invariant. When \( n = 2 \), the Sobolev inequality implies

\[
\left( \int_M h^{\frac{4}{n-2}} \, dV \right)^{\frac{n-2}{n}} \leq C(\kappa) \text{Vol}(M)^{\frac{n-2}{n}} \left( \int_M |\nabla^M h|^2 \, dV + \text{Vol}(M)^{-1} \int_M h^2 \, dV \right), \tag{2}
\]

for any \( \kappa > 2 \). Because both \( \bar{H} \) and \( \text{Vol}(M) \) are uniformly bounded in our cases, we thus omit the dependency of the constants on \( \sup |\bar{H}|^2 \text{Vol}(M)^{\frac{2}{n}} \).

By the above inequality, we have the following estimate:

**Lemma 5** Suppose \( u \) is a positive sub-solution of the equation \( \Delta_M u \geq gu \) on a closed manifold \( M \), where \( g \) is a \( L^1 \) function satisfying the estimate

\[ \|g\|_{L^1} \leq \tilde{c} \text{Vol}(M)^{\frac{2}{2} - \frac{2}{n}} \]

for some \( r > n \). Then \( \|u\|_{0,M} \leq C_p \text{Vol}(M)^{-\frac{2}{p}} \|u\|_{L^p} \)

for \( p > 0 \). The constant \( C_p \) depends on \( n, r, \tilde{c} \) and \( p \).

**Proof.** Multiply both sides of the inequality by \( u^{q-1} \), \( q \geq 2 \), and then integrate over \( M \). One thus has

\[
\int_M u^{q-1} \Delta_M u \, dV \geq \int_M gu^q \, dV,
\]

for some
or
\[-(q - 1) \int_M u^{q - 2} |\nabla^M u|^2 dV \geq \int_M g u^q dV.\]

By rewriting the left hand side and using Hölder inequality, this leads to
\[
\frac{4(q - 1)}{q^2} \int_M |\nabla^M u|^2 dV \leq \|g\|_{L^\frac{n}{2}} \|u^q\|_{L^\frac{2n}{n-2}}.
\]

Plug this into the Sobolev inequality. For \(n > 2\), one gets
\[
(f_M u^{\frac{\kappa}{2} - \frac{2n}{n-2}} dV)^{\frac{n-2}{n}} \leq C(n) (f_M |\nabla^M u|^2 dV + \text{Vol}(M)^{-\frac{\kappa}{2}} \int_M u^q dV)
\]
\[
\leq C(n) (cq \|g\|_{L^\frac{n}{2}} \|u^q\|_{L^\frac{2n}{n-2}} + \text{Vol}(M)^{\frac{2}{2} - \frac{\kappa}{n}} \|u^q\|_{L^\frac{2n}{n-2}})
\]
\[
\leq Cq \text{Vol}(M)^{\frac{2}{2} - \frac{\kappa}{n}} \|u^q\|_{L^\frac{2n}{n-2}}.
\]

The constant \(C\) depends on \(n\) and \(\kappa\). For \(n = 2\), we can choose \(\kappa = \frac{r+2}{2}\) and similarly get
\[
(f_M u^{\frac{\kappa}{2} - \frac{2n}{n-2}} dV)^{\frac{n-2}{n}} \leq Cq \text{Vol}(M)^{\frac{2}{2} - \frac{\kappa}{2}} \|u^q\|_{L^\frac{2n}{n-2}}.
\]

The constant \(C\) depends on \(\kappa\) and \(\bar{c}\). Denote \(\hat{n} = n\) for \(n > 2\) and \(\hat{n} = \kappa\) for \(n = 2\). Thus one has
\[
(f_M u^{\hat{n} - \frac{2n}{n-2}} dV)^{\frac{n-2}{n}} \leq (Cq \text{Vol}(M)^{\frac{2}{2} - \frac{\hat{n}}{2}})^{\frac{1}{\hat{n}}} \|u^q\|_{L^\frac{2n}{n-2}}.
\]

Denote \(\Psi(x) = (f_M u^x dV)^{\frac{1}{2}}\). Then the inequality can be written as
\[
\Psi(qk) \leq (Cq \text{Vol}(M)^{\frac{2}{2} - \frac{\hat{n}}{\hat{n}}})^{\frac{1}{\hat{n}}} \Psi(qs),
\]

where \(k = \frac{\hat{n}}{\hat{n}-2}\) and \(s = \frac{r}{r-2}\). Because \(r\) is greater than \(\hat{n}\), thus \(\gamma = \frac{k}{s}\) is greater than one, and
\[
\Psi(\gamma x) \leq (C^{\frac{2}{s}} \text{Vol}(M)^{\frac{2}{2} - \frac{\hat{n}}{\hat{n}}})^{\frac{1}{s}} \Psi(x),
\]

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for any $x \geq 2s$. Choose $x = \gamma^{m-1}p$ and $p \geq 2s$. One then has
\[
\Psi(p\gamma^{m}) \leq (C \frac{\gamma^{m-1}}{s} \text{Vol}(M)^{\frac{3}{2} - \frac{3}{2}}) \gamma^{m-t} \Psi(p\gamma^{m-1})
\]
\[
\leq (\frac{\gamma^{m}}{s} \text{Vol}(M)^{\frac{3}{2} - \frac{3}{2}}) \frac{\gamma^{m-1}}{s} \gamma^{\frac{m-1}{s}} \gamma^{\frac{m-1}{s}} \Psi(p).
\]

Let $m$ go to infinity and notice that
\[
\sum_{i=0}^{\infty} \frac{1}{\gamma^i} = \frac{k}{k-s} \quad \text{and} \quad -\frac{1}{k} + \frac{1}{s} = \frac{2}{n} - \frac{2}{r}.
\]

One then gets
\[
\|u\|_{0,M} \leq C \text{Vol}(M)^{-\frac{1}{k}} \|u\|_{L^p}, \quad \text{for } p \geq 2s. \quad (3)
\]

The constant $C$ depends on $\tilde{n}$, $r$ and $\tilde{c}$. For general $p$, first recall that one has
\[
(f_{\tilde{M}} u^{qk} dV)^{\frac{1}{k}} \leq Cq \text{Vol}(M)^{\frac{3}{2} - \frac{3}{2}} (f_{\tilde{M}} u^{qk} dV)^{\frac{1}{k}}
\]
\[
= Cq \text{Vol}(M)^{\frac{1}{2} - \frac{1}{k}} (f_{\tilde{M}} u^{qk} dV)^{\frac{1}{k}}.
\]

Therefore,
\[
\text{Vol}(M)^{-\frac{1}{k}} (f_{\tilde{M}} u^{qk} dV)^{\frac{1}{k}}
\]
\[
\leq (Cq)^{\frac{1}{k}} [\text{Vol}(M)^{-\frac{1}{k}} (f_{\tilde{M}} u^{qk(1-\varepsilon)\lambda} dV)^{\frac{1}{k}} \text{Vol}(M)^{-\frac{1}{k}} (f_{\tilde{M}} u^{qk\varepsilon} dV)^{\frac{1}{k}}]^{\frac{1}{k}}
\]

where $1 > \varepsilon > 0$ and $\frac{1}{\lambda} + \frac{1}{\varepsilon} = 1$. If we choose $\lambda$ satisfying $q\varepsilon(1-\varepsilon)\lambda = qk$, then it follows that
\[
\text{Vol}(M)^{-\frac{1}{k}} (f_{\tilde{M}} u^{qk} dV)^{\frac{1}{k}} \leq (Cq)^{\frac{1}{k}} \text{Vol}(M)^{-\frac{1}{k}} (f_{\tilde{M}} u^{qk\varepsilon} dV)^{\frac{1}{k}}.
\]

That is,
\[
\text{Vol}(M)^{-\frac{1}{k}} (f_{\tilde{M}} u^{qk} dV)^{\frac{1}{k}} \leq (Cq)^{\frac{1}{k}} \text{Vol}(M)^{-\frac{1}{k}} (f_{\tilde{M}} u^{qk\varepsilon} dV)^{\frac{1}{k}}.
\]
Let $q = 2$ and $p = 2se\mu = \frac{2se}{k-\varepsilon+\mu}$, then

$$Vol(M)^{-\frac{1}{p}} \left( \int_M u^{2k} dV \right)^{\frac{1}{2k}} \leq (2C)^{\frac{1}{2}} Vol(M)^{-\frac{1}{p}} \left( \int_M u^p dV \right)^{\frac{1}{p}}. \quad (4)$$

By varying $\varepsilon$, we can choose $p$ to be any positive number. Combine (3) and (4), and one gets

$$\|u\|_{0,M} \leq C_p Vol(M)^{-\frac{1}{p}} \|u\|_{L^p}, \quad \text{for } p > 0.$$ 

Q.E.D.

**Remark:** The constant on the right hand side of the inequality (1) or (2) is called the Sobolev constant on $M$. The quantity can be defined in a general Riemannian manifold, which is not necessarily a submanifold of $\mathbb{R}^4$. We only use (1) or (2) to derive the estimate. Thus the lemma holds in general and the constant $C_p$ depends on the Sobolev constant on $M$, $r$, $\varepsilon$ and $p$.

From Lemma 5, We can get the following supremum estimate:

**Theorem 5** Suppose that $u$ is a $W^{1,2}$ weak solution for $\Delta_M u = f$ on a closed Riemannian manifold $M$, where $f$ satisfies $\|f\|_{L^2} < \infty$, for some $r > n$. Then

$$\|u\|_{0,M} \leq C \left( Vol(M)^{-\frac{1}{2}} \|u\|_{L^2} + Vol(M)^{\frac{1}{2} - \frac{2}{r}} \|f\|_{L^\frac{2}{r}} \right),$$

where $C$ depends only on the Sobolev constant on $M$ and $r$.

**Proof.** Define $\beta = Vol(M)^{\frac{1}{2} - \frac{2}{r}} \|f\|_{L^\frac{2}{r}}$ and $\xi = \frac{1}{2} \{ \frac{\mu}{\beta} \}^2 + 1$. It is easy to see that $\xi$ is a weak solution for

$$\Delta_M \xi \geq \frac{f u}{\beta \beta} \overset{\xi}{=} \frac{f u}{\beta \beta} \frac{1}{\xi} \xi.$$
Denote \( g = \frac{f}{\beta} \frac{u}{\beta} \frac{1}{\xi} \). Because \( \left| \frac{u}{\beta \xi} \right| < 2 \), one then has
\[
\| g \|_{L^2} \leq \frac{2}{\beta} \| f \|_{L^2} \leq 2 \text{Vol}(M)^{\frac{3}{2} - \frac{1}{2}}.
\]
By Lemma 5, it follows that
\[
\xi \leq C \text{Vol}(M)^{-1} \| \xi \|_{L^1}.
\]
Hence
\[
\left| \frac{u}{\beta} \right|^2 \leq C \text{Vol}(M)^{-1} \left[ \frac{1}{2} \left( \frac{u}{\beta} \right)^2 \right]_{L^1} + \text{Vol}(M).
\]
Therefore,
\[
|u| \leq C \beta \sqrt{1 + \text{Vol}(M)^{-1} \| \left( \frac{u}{\beta} \right)^2 \|_{L^1}}
\]
\[
\leq C \sqrt{\beta^2 + \text{Vol}(M)^{-1} \| u^2 \|_{L^1}}
\]
\[
\leq C (\beta + \text{Vol}(M)^{-\frac{1}{2}} \| u \|_{L^2})
\]
\[
= C \left( \text{Vol}(M)^{\frac{3}{2} - \frac{1}{2}} \| f \|_{L^2} \right) + \text{Vol}(M)^{-\frac{1}{2}} \| u \|_{L^2}).
\]
Again, the constant \( C \) may change slightly in different places.

Q.E.D.

Suppose that \( M_\alpha \) and \( \rho \) are as defined in section 3 and section 5. We need the following estimate in weighted norm to prove the main theorem.

Lemma 6 [9] Suppose that \( u \) is a \( C^{2,\beta} \) function on \( M_\alpha \), \( 0 < \beta < 1 \), which satisfies \( f_{M_\alpha} u \, dV = 0 \). Then one has \( \| u \|_{0, M_\alpha} \leq C \varepsilon^{-\nu} \| \rho^2 \Delta_{M_\alpha} u \|_{C^0, \alpha(M_\alpha)} \) for \( \alpha \) small enough, where \( \nu \) is any positive number. The constant \( C \) depends on \( \nu \), but is independent of \( \alpha \).
Proof. Assume that the Lemma does not hold. Then there exists a sequence $\alpha_j \to 0$, its corresponding $\varepsilon_j, \rho_j$, and $u_j \in C^{2,\beta}(M_{\alpha_j})$ which satisfies $\int_{M_{\alpha_j}} u_j \, dV = 0$ and

$$
\|u_j\|_{0, M_{\alpha_j}} \geq j \varepsilon_j \|\rho_j^2 \Delta_{M_{\alpha_j}} u_j\|_{C^{0,\beta}_{\rho_j}(M_{\alpha_j})}.
$$

We can normalize $u_j$ such that $\|u_j\|_{0, M_{\alpha_j}} = 1$. It then follows that

$$
\|\rho_j^2 \Delta_{M_{\alpha_j}} u_j\|_{C^{0,\beta}_{\rho_j}(M_{\alpha_j})} \leq \frac{1}{j} \varepsilon_j.
$$

On the other hand, by Theorem 4 we have

$$
\|u_j\|_{0, M_{\alpha_j}} \leq C \left[ \text{Vol}(M_{\alpha_j})^{-\frac{1}{2}} \|u_j\|_{L^2} + \text{Vol}(M_{\alpha_j})^{\frac{5}{4} - \frac{1}{p}} \|\Delta_{M_{\alpha_j}} u_j\|_{L^p} \right].
$$

Because $u_j$ satisfies $\int_{M_{\alpha_j}} u_j \, dV = 0$, one has

$$
\lambda_1(M_{\alpha_j}) \int_{M_{\alpha_j}} u_j^2 \, dV \leq - \int_{M_{\alpha_j}} \langle \Delta_{M_{\alpha_j}} u_j, u_j \rangle \, dV \leq \|\Delta_{M_{\alpha_j}} u_j\|_{L^1}.
$$

Remember that the Sobolev constant on $M_{\alpha_j}$ is bounded uniformly, the volume $\text{Vol}(M_{\alpha_j})$ is bounded uniformly from above and below, and $\lambda_1(M_{\alpha_j})$ is bounded below by $\frac{1}{4} \lambda_1(L)$. Thus when $r$ satisfies $-r + \frac{5r}{2} \geq -n$,

$$
1 = \|u_j\|_{0, M_{\alpha_j}} \leq C_r \left( \|\Delta_{M_{\alpha_j}} u_j\|_{L^1} + \|\Delta_{M_{\alpha_j}} u_j\|_{L^\frac{5}{2}} \right)
$$

$$
\leq C_r \left[ \int_{M_{\alpha_j}} \rho_j^{-r}(\rho_j^2 \Delta_{M_{\alpha_j}} u_j)^\frac{5}{2} \, dV \right]^\frac{2}{5}
$$

$$
\leq C_r \left[ \int_{M_{\alpha_j}} \rho_j^{-r} \left( \frac{1}{j} \varepsilon_j \right)^\frac{5}{2} \, dV \right]^\frac{2}{5}
$$

$$
\leq C_r \left[ \int_{M_{\alpha_j}} \rho_j^{-r + \frac{5r}{2}} \, dV \right]^\frac{2}{5}
$$

$$
\leq \frac{C_r}{j}.
$$

The constant $C_r$ may change in different places. Given $\nu > 0$, we first find $r$ which satisfies $-r + \frac{5r}{2} \geq -n$. Because the constant $C_r$ is independent of $j$,
the above inequality then leads to a contradiction. Hence the lemma must hold.

Q.E.D.

References


