Normalised quadratic controls for a class of bilinear systems

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Abstract: Normalised quadratic control, a modification of the conventional quadratic control method, has recently been proposed to exponentially stabilise a homogeneous bilinear system whose open-loop eigenvalues all fall on the imaginary axis. The normalised quadratic control method is now extended to a broader class of bilinear systems whose open-loop eigenvalues may fall to the left or right of the imaginary axis. It is shown that for bilinear systems that are already open-loop stable, the normalised quadratic control can add to the system an extra exponential decay rate. For open-loop unstable bilinear systems, the normalised quadratic control can stabilise the system if the extra exponential decay rate provided by the control is larger than the open-loop system's growth rate.

1 Introduction

Consider a homogeneous bilinear system:

\[ \dot{x}(t) = Ax(t) + u(t)Nx(t), \quad x(0) = x_0 \]  

where \( x(t) \in \mathbb{R}^n \) is the system state vector, \( u(t) \) is a scalar control input, and \( A \in \mathbb{R}^{n \times n} \) and \( N \in \mathbb{R}^{n \times n} \) are constant square matrices. It is assumed that the open-loop system matrix \( A \) has only pure imaginary eigenvalues. In this case [1], there exists a positive definite matrix \( P \) such that:

\[ A^TP + PA = 0 \]  

(2)

Under this neutrally stable assumption, the pair \((A, N)\) satisfies the following controllability rank condition [2, 3]: for some positive integer \( m \):

\[ \text{rank}[ad^k(A, N)x_0, ad^{k+1}(A, N)x_0, \ldots, ad^{m}(A, N)x_0] = n \]  

(3)

for any nonzero \( x_0 \) in \( \mathbb{R}^n \), where:

\[ ad^k(A, N) = N, \quad ad^{k+1}(A, N) = A \times ad^k(A, N) \]

\[ = \ldots = \ldots \]  

In the literature, two different control methods have been proposed for an open-loop neutrally stable system (1). The first control method is quadratic feedback control [4-6], which ensures global asymptotic stability of the closed-loop system. The second control method is a modification of the first one, called normalised quadratic feedback control:

\[ u(t) = -\gamma e^{T}(t)PNe(t) \quad e(t) = \frac{x(t)}{\|x(t)\|} \]  

(4)

however, the matrix \( P \) is now obtained from the following modified Lyapunov equation instead of (2):

\[ (A + \rho I)^TP + P(A + \rho I) = -Q \]  

(5)

where the plus sign is for a stable \( A \), the minus sign is for neutrally stable and unstable \( A \), \( P \) and \( Q \) are two positive definite matrices, and \( \rho > 0 \) is defined by:

\[ \rho = \begin{cases} -\max\{\Re(\lambda_i(A))\} - \varepsilon & \text{for stable } A \\ \varepsilon & \text{for neutrally stable } A \\ \max\{\Re(\lambda_i(A))\} + \varepsilon & \text{for unstable } A \end{cases} \]  

(6)

in which \( \varepsilon > 0 \) is an arbitrarily small number. When \( \varepsilon \) is sufficiently small and for the case with stable \( A \), \( \rho \) characterises the decay rate of the open-loop system. For the case with unstable \( A \), \( \rho \) characterises the growth rate of the open-loop system.

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Before analysing the normalised control (4), one needs the following assumption: for some positive integer $m$:

$$\text{rank} [R_0 x_0, R_1 x_0, \ldots, R_m x_0] = n \quad (7)$$

for any nonzero $x_0$ in $\mathbb{R}^n$, where $R_k$ is defined recursively as $R_0 = PN$, $R_k = A^T R_{k-1} + R_{k-1} A, \ k = 1, 2, 3 \ldots$

It is speculated that the rank condition (7) may be deduced from the rank condition (3) for any stable or unstable matrix. Further research will be required to verify this speculation.

**Lemma 1:** Consider the closed-loop system (1) and (4). If the rank condition (7) is satisfied, and the system state $x(t)$ satisfies:

$$x'(t) PN x(t) = 0 \quad \forall t \in [kT, kT + T) \quad (8)$$

for some $T > 0$, then $x(kT) = 0$.

**Proof:** If $x'(t) PN x(t)$ is identically zero over the time span $[kT, kT + T)$, so is the control input (4). The system (1) then becomes open-loop; therefore:

$$x(t) = e^{A^T \tau} x(kT) \quad \forall t \in [kT, kT + T)$$

Substituting the above equation into (8), and taking consecutively the time derivatives of the equation, one obtains

$$x'(kT) R A x(kT) = 0, \quad i = 1, 2, 3, \ldots, m,$n or in matrix form,

$$x'(kT) [R_c x(kT) | R_x x(kT) | \ldots | R_m x(kT)] = 0$$

The above equation shows that $x(kT)$ is orthogonal to all vectors $R_c x(kT)$. Since $R_c x(kT)$, $i = 1, 2, \ldots, m$, span the whole space according to assumption (7), $x(kT)$ must be the null vector.

Further, to quantify the stabilising effect of the normalised control (4), one needs the following definition. Given a time interval length $T > 0$, define a function $f(t): S \rightarrow \mathbb{R}^+ \cup \{0\}$ for the controlled system (1) and (4), where $S$ is the unit sphere in $\mathbb{R}^n$:

$$\beta(e_i(kT)) \triangleq \frac{1}{\sigma_p T} \int_{kT}^{kT+T} [e_i'(t) PN e_i(t)]^2 \, dt \quad e_i(kT) \in S \quad (9)$$

in which $\gamma$ is the control gain in (4), and $\sigma_p$, the maximum singular value of $P$ in (5). The following lemma shows that the rank condition (7) guarantees that $\beta(e_i(kT))$, which is a measure of the exponential stabilising effect of the normalised control (4) over a time span $T$ (see remark 3 after theorem 2), is always nonzero for all $e_i(kT)$ on the unit sphere $S$.

**Lemma 2:** If the system (1) satisfies the rank condition (7), there exists some positive constant $\beta^*$ such that for all integer $k$:

$$\inf_{e_i(kT) \in S} \beta(e_i(kT)) = \beta^* > 0 \quad (10)$$

**Proof:** Since the integrand in (9) is non-negative, $\beta(\cdot)$ must also be non-negative. To show that $\beta(\cdot)$ is nonzero (hence, positive) for all $e_i(kT) \in S$, one can use the following contradiction argument. Assume that $\beta(e_i(kT))$ is zero for some $e_i(kT) \in S$. Based on the definition of $\beta(\cdot)$ in (9), one has $e_i'(t) PN e_i(t) = 0$, for all $t \in [kT, kT + T)$. In other words:

$$x'(t) PN x(t) = 0 \quad \forall t \in [kT, kT + T)$$

Invoking lemma 1, the above equation implies $x(kT) = 0$, which contradicts the fact that $e_i(kT) \in S$. Therefore, $\beta(e_i(kT))$ must be positive for all $e_i(kT) \in S$. Furthermore, since the unit sphere $S$ is compact, the infimum of $\beta(e_i(kT))$ over $S$, denoted by $\beta^*$, must also be positive according to theorem 4.4.1 in [8]. Finally, since the dynamics of the closed-loop system (1) and (4) are the same on all different time intervals $[kT, kT + T)$, the lower bound $\beta^*$ must be uniform with respect to $k$. This finished the proof of (10).

One can now present the stability results for the normalised control (4), for the open-loop stable system (1), and then for the open-loop unstable system.

**Theorem 1:** Consider an open-loop stable bilinear system (1). With the continuous normalised quadratic control (4), the closed-loop system is exponentially stable with the following decay rate:

$$\|x(kT + T)\| \leq \frac{\sigma_Q}{\sigma_p} \times \exp(-\beta^* + \frac{\sigma_Q}{2\sigma_p} + \rho) \times \|x(kT)\| \quad (11)$$

**Proof:** Define a Lyapunov function candidate $V(t) = x'(t) Q x(t)$, where $P$ is from the Lyapunov equation (5) with stable $A$. Take the time derivative of the Lyapunov function along the closed-loop trajectory (1), (4) and (5):

$$V(t) = -x'(t) Q x(t) - 2x'(t) P x(t)$$

Integrating the inequality over the time span $[kT, kT + T)$ leads to:

$$\ln \frac{V(kT + T)}{V(kT)} \leq -\frac{\sigma_Q}{\sigma_p} T - 2\rho \times T$$

where the last inequality results from (10) of lemma 2. Equation (11) can then be concluded from the last inequality and from the definition of $V(kT)$.

**Remark 1:** Before interpreting the result (11) of theorem 1, note that $A \neq \mu I$ in the Lyapunov equation (4) is stable but close to neutrally stable for small $\epsilon$ (it is neutrally stable if $\epsilon = 0$). According to lemma 5 in the Appendix, one has:

$$\lim_{\epsilon \rightarrow 0} \frac{\sigma_Q}{\sigma_p} = 0$$

and hence for small $\epsilon$ in (6), one can neglect the term $\sigma_Q/2\sigma_p$ in the exponent of (11). Therefore, for an open-loop stable bilinear system (1), the decay rate resulting from the normalised quadratic control is given by:

$$\left\|x(kT + T)\right\| \leq \frac{\sigma_Q}{\sigma_p} \times e^{-\beta^* + \rho} \times \left\|x(kT)\right\| \quad (12)$$

Recall from the statement below (6) that $\rho$ characterises the decay rate of the open-loop stable bilinear system. Equation (12) then shows that the normalised control has the effect of providing an extra exponential decay rate by an exponent of $\beta^*$ for an open-loop stable bilinear system.

The second stability theorem is for open-loop unstable systems, whose proof parallels that of theorem 1, and hence is omitted.
Theorem 2: Consider an open-loop-unstable bilinear system (1). The continuous normalised quadratic control (4) can exponentially stabilise the closed-loop system if:

\[ \beta^* + \frac{\sigma_P}{2\sigma_P} > \psi \]

Furthermore, the closed-loop decay rate is given by:

\[ \|x(kT + T)\| \leq \frac{\sigma_P}{2\sigma_P} e^{-\psi} - \psi T \times \|x(kT)\| \]

(13)

Remark 2: As stated in remark 1, for small \( \varepsilon \) in (6), one can neglect the term \( \frac{\sigma_P}{2\sigma_P} \) in the exponent of (13), which then becomes:

\[ \|x(kT + T)\| \leq e^{-\psi} - \psi T \times \|x(kT)\| \]

(14)

Recall again that \( \psi \) in (4) characterises the exponential growth rate of the open-loop unstable bilinear system. Hence, (14) states that an open-loop unstable bilinear system can be stabilised if the exponential decay rate \( \beta^* \) provided by the normalised control is larger than the growth rate \( \psi \) of the open-loop system.

Remark 3: Based on the observations in remarks 1 and 2, one concludes that \( \beta^* \) is actually a measure of the size in the exponent of the normalised control (4). Metaphorically speaking, the normalised control has an equivalent effect of, as in linear system control, shifting the open-loop eigenvalues to the left by a distance of \( \beta^* \). Note that \( \beta^* \) is a function of the control gain \( \gamma \). In practical use of the normalised control, one needs to simulate control with various control gains in order to find out the optimal gain \( \gamma \) to maximise \( \beta^* \).

3 Discontinuous Normalised Quadratic Control

In the preceding Section, it was shown that the normalised control has a stabilising effect if the bilinear system satisfies the rank condition (7). However, this rank condition depends on the matrix \( P \) in (5) and hence on the matrix \( Q \), which is chosen by the control designer. This suggests that the designer's choice of controller parameters may affect whether or not the specific control design has a stabilising effect. To avoid such complexity, a different normalised control design will be developed in this Section, and the new normalised control is guaranteed to have an exponential stabilising effect under the original rank condition (3) in [7], which is independent of controller parameters.

For the second normalised control design, apply the following coordinate transformation to the bilinear system (1):

\[ x(t) = e^{\psi(t-t')} z(t) \quad t \in [kT, kT + T) \]

(15)

where \( z(t) \) is the transformed state, and the transformation matrix is the fundamental matrix [9] of the open-loop system (1). Note that the transformed state \( z(t) \) is defined only on a finite time interval \([kT, kT + T)\); nevertheless, this will not affect the asymptotic stability analysis presented below.

With the transformation (15), the governing equation of the transformed state becomes:

\[ z(t) = u(t) G_d(t) z(t) \quad G_d(t) = e^{-\psi(t-t')} N e^{\psi(t-t')} \quad t \in [kT, kT + T) \]

(16)

The above transformed system becomes open-loop neutrally stable, similar to the case treated in [7] except that now the neutrally stable system (16) contains time-varying parameters. However, one can still follow the control design principle in [7] to construct the following normalised quadratic feedback control:

\[ u(t) = -\gamma e^T(t) G_d(t) z(t) \quad e(t) = \frac{z(t)}{\|z(t)\|} \quad t \in [kT, kT + T) \]

(17)

where \( \gamma \) is a positive control gain. Note that this second normalised control is discontinuous at the time instants \( t = kT \) because of the reset of the transformed state at those instants.

The stability analysis of the second normalised control (17) parallels that of the first normalised control (4); therefore, proofs of the following lemmas are omitted for brevity.

Lemma 3: If the system (1) satisfies the rank condition (3), and there exist constant vectors \( z_0 \) such that \( z_0^T G_d(t) z_0 = 0 \) for all \( t \in [kT, kT + T) \), then \( z_0 \) must be the null vector.

To quantify the stabilising effect of the new normalised control (17), define a function \( \alpha(x) : S \rightarrow \mathbb{R}^+ \cup \{0\} \), where \( S \) is the unit sphere in \( \mathbb{R}^n \), for the transformed closed-loop system (16) and (17):

\[ \alpha(z(t)) = \frac{1}{T} \int_0^T [e^T(t) G_d(t) e(t)]^2 dt \quad e(t) \in S \]

(18)

Lemma 4: If the bilinear system (16) satisfies the rank condition (3), there exists some positive constant \( \alpha^* \) such that for all positive integer \( k \):

\[ \inf_{e(t) \in S} \alpha(z(kT)) = \alpha^* > 0 \]

(19)

One can now prove the global exponential stability of the bilinear system (1) under the second normalised control (17).

Theorem 3: Consider the bilinear system (1) and the normalised control (17). If:

\[ e^{\psi T} > \|x^T\| \]

the closed-loop system is exponentially stable with the following decay rate:

\[ \|x(kT + T)\| \leq \pi \|x(kT)\| \]

where \( \pi = \|e^{\psi T}\| / e^{\psi T} < 1 \)

(20)

Proof: In this proof, one will first estimate the variation of the transformed state \( \|z(t)\| \) over the time interval \([kT, kT + T)\), and then utilise the obtained result to estimate the variation of the original state \( \|x(t)\| \).

To do so, first define:

\[ V_4(t) = \frac{1}{2} \|z(t)\|^2 \quad t \in [kT, (k + 1)T) \]

and take the time derivative of \( V_4(t) \) along (16) and (17):

\[ V_4(t) = z_0^T(t) G_d(t) z_0(t) u(t) = -2\gamma |e^T(t) G_d(t) e(t)|^2 V_4(t) \]

Integrating the above equation from \( kT \) to \((k + 1)T \) gives:

\[ V_4(kT + T) - V_4(kT) = -2\gamma \int_{kT}^{(k + 1)T} |e^T(t) G_d(t) e(t)|^2 dt \leq -2\alpha^* T \]

where the last inequality is obtained via lemma 4. Re-arranging the equation gives:

\[ V_4(kT + T) \leq e^{-2\alpha^* T} V_4(kT) \]

or equivalently:

\[ \|z_0(kT + T)\| \leq e^{-\alpha^* T} \|z_0(kT)\| \]

(22)

Next, to find out how the original system state \( \|x(kT)\| \) varies, recall from (15) that:

\[ x(kT) = z_0(kT) \quad x(kT + T) = e^{\psi T} z_0(kT + T) \]

(23)
Taking the norm of $x(kT + T)$ in (23), and using (22), one can then prove (21):

$$\|x(kT + T)\| \leq \frac{\|e^{AT}\|}{e^{\gamma T}} \times \|x(kT)\| = \frac{\|e^{AT}\|}{e^{\gamma T}} \times \|x(kT)\|$$

where the last equality results from the first identity in (23).

Finally, the assumption (20) guarantees that the closed-loop system is exponentially stable. \(\Box\)

Remark 4: Given any system matrix $A$, there exists a positive constant $m$ such that:

$$\|e^{AT}\| \leq m e^{\gamma T}$$

where the minus sign is for stable $A$, and the plus sign is for unstable $A$, and $\gamma$ as defined in (6). With (24), (21) becomes, for stable $A$:

$$\|x(kT + T)\| \leq m e^{\gamma T} \times \|x(kT)\|$$

and for unstable $A$:

$$\|x(kT + T)\| \leq m e^{\gamma T} \times \|x(kT)\|$$

Equation (25), which is similar to (12) in the preceding Section, suggests that when the system (1) is open-loop stable, the second normalised control (17) can add to the open-loop decay rate $\gamma$ an extra decay rate $\alpha^*$. Equation (26), which is similar to (14) in the preceding Section, suggests that when the system (1) is open-loop unstable, the second normalised control (17) can still stabilise the closed-loop system if the decay rate $\alpha^*$ provided by the control is larger than the open-loop growth rate $\gamma$. In conclusion, $\alpha^*$ defined in (19) plays the same role as $\beta^*$ in the preceding Section; that is, $\alpha^*$ is a measure of the exponentially stabilising effect for the second normalised control (17).

Example 1: Consider an open-loop stable bilinear system (1) with:

$$A = \begin{bmatrix} -1 & -5 \\ 4 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the initial condition $x^T(0) = [5, -4]$. Fig. 1 shows the open-loop response (the dash line) and the closed-loop response (the solid line) of the system under the second normalised quadratic control (17) with $y = 1$ and $T = 1$. It is seen that the normalised quadratic control makes the decay rate three times faster than the open-loop. Fig. 2 depicts the normalised control signal.

Example 2: Consider an open-loop unstable bilinear system (1) with:

$$A = \begin{bmatrix} 0.5 & 1 \\ -1 & 0.5 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $x^T(0) = [5, -4]$. Fig. 3 shows that the normalised quadratic control (17) with $y = 2$ and $T = 1$ effectively stabilises the closed-loop system. Fig. 4 shows the control signal.
signal, which seems to asymptotically approach a constant $u^* = -1.39$. However, one can verify that a constant control design $u(t) = u^*$ will not stabilise the bilinear system.

4 Conclusions

This paper studies two normalised quadratic controls (NQC) for a homogeneous bilinear system which satisfies the controllability rank condition. It is shown that the NQC can provide an extra exponential decay rate if the open-loop system is already stable, and can stabilise an open-loop unstable system if the extra exponential decay rate provided by the control is larger than the growth rate of the open-loop system. However, the theoretical analysis in this paper is more qualitative than quantitative in the sense that the extra decay rate provided by NQC, namely $\beta$ in (10) or $\alpha$ in (19), is difficult to calculate. In practical use of NQC, one must simulate control with different control gains $y$ to find out if the NQC can stabilise a particular open-loop unstable system, and, if it can, what is the optimal control gain $y$ for achieving the fastest decay rate.

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6 References

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7 Appendix

Lemma 5: Given the Lyapunov equation (5), one has

$$\lim_{t \to 0} \frac{\sigma}{\tau} = 0$$

where $\sigma$ denotes the maximum singular value of the matrix $P$, and $\tau$ the minimum singular value of $Q$.

Proof: Without loss of generality, consider the case where $Q = I$ in (5). Therefore:

$$\sigma = 1$$

$$\tau = \max_{\|y\| = 1} \int_0^\infty y_0^T e^{(A + B) t} y_0 \, dt$$

$$= \max_{\|y\| = 1} \int_0^\infty \|y(t)\|^2 \, dt$$

where $y(t) = e^{(A + B) t} y_0$ denotes the solution of the differential equation $\dot{y} = (A + B) y$ subject to the initial condition $y(0) = y_0$. By the definition of $\rho$ in (6), the matrix $A + B$ is stable. Denote its eigenvalues by $\lambda_i = -\alpha + j \beta_i$ with $\alpha_i \geq \alpha$, $i = 1, 2, \ldots, n$. Then, according to (6):

$$\sigma = e$$

Asymptotically, the time history of $y(t)$ is determined by the dominant mode $i = 1$; hence, $\lim_{t \to \infty} y(t) = c_1 e^{-\alpha t} \sin(\beta_1 t + \phi)$ for some constants $c_1$ and $\phi$. Substituting $y(t)$ into (28), and using (29), one can show that:

$$\sigma = O(\frac{1}{\tau})$$

The lemma is then proved by combining (27) and (30).