A4 Group and Tri-bimaximal Neutrino Mixing—A Renormalizable Model

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The tetrahedron A4 group has been widely used in studying neutrino mixing. It provides a natural framework of model building for the tri-bimaximal mixing matrix. In this class of models, it is necessary to have two Higgs fields, χ and χ', transforming under A4 as 3 with one of them having vacuum expectation values for the three components to be equal and another having only one of the components to be non-zero.

These specific vev structures require separating χ and χ' from communicating with each other. The clash of the different vev structures for χ and χ' is the so called sequestering problem. In this work, I show that it is possible to construct renormalizable supersymmetric models producing the tri-bimaximal neutrino mixing with no sequestering problem.

The current data from neutrino oscillation experiments[1] can be described by three neutrino mixing. The mixing matrix V can be well fitted by the tri-bimaximal mixing of the form[2]

\[ V_{tri-bi} = \begin{pmatrix} 2 & 1 & 0 \\ \sqrt{6}/\sqrt{3} & \sqrt{3}/\sqrt{2} & -1/\sqrt{2} \\ -\sqrt{6}/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} P. \]  (1)

Here \( P = \text{Diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) \) is a Majorana phase matrix. Since an overall phase does not play a role in any physical process, only two of the \( \alpha_{1,2,3} \) are physically independent.

The tri-bimaximal form for neutrino mixing was first proposed by Harrison, Perkins and Scott[2], and further studied by Xing[2]. Also independently proposed by He and Zee[2]. Many theoretical efforts have been made to produce such a mixing pattern. Among them theories based on A4 symmetry provide some interesting examples[3–5]. Most of the attempts made in the literature assumed certain vacuum expectation value (vev) structures for Higgs fields without specific renormalizable models to realize them. Attempts to build renormalizable models have been made in Refs.[4,5]. Here I construct a realistic renormalizable model with supersymmetry (SUSY) which produces the tri-bimaximal mixing.

In addition to the standard \( SU(3)_C \times SU(2)_L \times U(1)_Y \) gauge symmetry, this model has additional global symmetries \( A_4 \times Z_4 \times Z_3 \times Z_2 \) acting on various fields. Under the global symmetries in order of \( A_4, Z_4, Z_3, \) and \( Z_2 \), the relevant lepton and Higgs fields transform as:

\[ L(3,1,0,1), \quad E^c(1,3,0,0,0), \quad H_u(1,3,2,0,1), \quad H_d(1,3,1,0,0), \quad \chi(3,0,1,0), \quad \chi'(3,0,1,0), \quad S(1,0,0,0), \quad S'(1,0,0,0), \quad S''(1,0,1,0), \quad S'''(1,2,0,0). \]

The A4 group is the tetrahedron group. It has 12 elements with 4 inequivalent representations \( 1, 1', 1'' \) and 3. The multiplication rules of these representations are \( 1 \times 1 = 1, 1 \times 1' = 1', 1 \times 1'' = 1'' \), \( 1' \times 1' = 1'' \), \( 1'' \times 1'' = 1' \). The 1, \( 1', 1'' \) and the two 3’s formed from two 3’s \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \) are given by

\[ 1 : a_1b_1 + a_2b_2 + a_3b_3, \]
\[ 1' : a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3, \]
\[ 1'' : a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3, \]
\[ 3_s : (a_2b_3 + a_3b_2, a_3b_1 + a_1b_3, a_1b_2 + a_2b_1), \]
\[ 3_a : (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1). \]

The \( Z_n \) charge \( N \) in the above are defined as \( \exp(2i\pi n/\ell). \)

The superpotential relevant to lepton masses are given by:

\[ W_Y = M_E E_i E_i^c S'' + f_a L_i E_i^c H_d + h_j^c E_i E_j^c \chi_k + \frac{1}{2} f_{ij} N_i N_j N_k + \frac{1}{2} f_{ijk} N_i N_j N_k \chi_k \]
\( f_{\nu L_i N^c_i H_u} \). The \( Z_2 \) is needed to prevent \( N^c \chi^2 \) term which induces mixing between \( N^c \) and \( \chi' \) with a non-zero vev for \( \chi' \) and causes problem in obtaining the tri-bimaximal mixing.

The vev's of \( (H_u) = v_u \), \( (H_d) = v_d \), \( (S') = v_{s'} \), \( (S'') = v_{s''} \), and \( \langle \chi(\chi') \rangle \) break the gauge symmetry and also the global symmetries. If \( (\chi_i) = x_i \) and \( (\chi'_i) = x'_i \) have the following form,

\[
x_1 = x_2 = x_3 = v_{\chi}, \ x'_1 = x'_3 = 0, \ x'_2 = v_{\chi'},
\]

the mass matrices \( M_{\nu E} \) and \( M_{\nu N} \) in the Lagrangian \( L = -(e, E) M_{\nu E}(e^\dagger, E^\dagger)^T - (\nu^c, N) M_{\nu N}(\nu, N^c)^T \) are given by

\[
M_{\nu E} = \begin{pmatrix} 0 & M_{\nu E c} \\ M_{\nu E c} & M_{EE c} \end{pmatrix},
M_{\nu N} = \begin{pmatrix} 0 & M_{\nu N c} \\ M_{\nu N c} & M_{NN c} \end{pmatrix},
\]

with

\[
M_{\nu E c} = \begin{pmatrix} f_{e v_d} & 0 & 0 \\ 0 & f_{e v_d} & 0 \\ 0 & 0 & f_{e v_d} \end{pmatrix},
M_{\nu E} = \begin{pmatrix} h_1^c v_{\chi} & h_2^c v_{\chi} & h_3^c v_{\chi} \\ h_1^c v_{\chi} & h_2^c \omega v_{\chi} & h_3^c \overline{\omega} v_{\chi} \\ h_1^c v_{\chi} & h_2^c \overline{\omega} v_{\chi} & h_3^c \omega v_{\chi} \end{pmatrix},
M_{EE c} = \begin{pmatrix} f_E v_{\nu'} & 0 & 0 \\ 0 & f_E v_{\nu'} & 0 \\ 0 & 0 & f_E v_{\nu'} \end{pmatrix},
M_{\nu N c} = M_{\nu N c} = \begin{pmatrix} f_{\nu u} & 0 & 0 \\ 0 & f_{\nu u} & 0 \\ 0 & 0 & f_{\nu u} \end{pmatrix},
M_{NN c} = \begin{pmatrix} f_{s' v_{s'}} & 0 & f_{s' v_{s'}} \\ 0 & f_{s' v_{s'}} & 0 \\ f_{s' v_{s'}} & 0 & f_{s' v_{s'}} \end{pmatrix}.
\]

The above results in the following form for the light lepton mass matrices,

\[
M_{\nu} = U_L \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_{\mu} & 0 \\ 0 & 0 & m_{\tau} \end{pmatrix},
M_{\nu}^{light} = m_0 \begin{pmatrix} 1 & 0 & x \\ 0 & 1 - x^2 & 0 \\ x & 0 & 1 \end{pmatrix},
U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},
\]

where the charged lepton masses \( m_{e, \mu, \tau} \) are give by

\[
m_i = \sqrt{3} \left( \frac{f_{e v_d v_{\chi}}/f_E v_{\nu'} h_i^c (1 + (h_i^c v_{\chi})^2)}{(f_E v_{\nu'})^2} \right)^{-1/2},
\]

and \( m_0 = f_{s' v_{s'}} (f_{s' v_{s'}}^2 - f_{\chi}^2 v_{\chi}^2) \), \( x = -f_{\chi} v_{\chi}/f_{s' v_{s'}} = |x| e^{i \psi} \).

Diagonalizing the lepton mass matrices, we obtain the neutrino mixing matrix given by eq.(1). The Majorana phase matrix \( P \) is given by: \( P = \text{Diag}(e^{-i \phi_1/2}, e^{-i(\phi_1+\phi_2)/2}, e^{-i(\phi_2+\pi)/2}) \) with \( \phi_1 = \arg(1 + x) \), \( \phi_2 = \arg(1 - x) \). The eigen-masses are given by \( m_1 = |m_0||1 + x| \), \( m_2 = |m_0||1 - x^2| \) and \( m_3 = |m_0||1 - x| \). Both normal and inverted neutrino mass hierarchies are allowed[4].

In order to obtain the tri-bimaximal mixing it is crucial to have the \( \chi \) and \( \chi' \) representation to have the specific vev structure in eq.(2). One needs to make sure that this vev structure is obtainable in a given model. In the following we demonstrate that the model proposed here can have the desired vev structure.

Non-zero vev’s of the Higgs break \( A_4 \), but left some residual symmetries. The vev of \( \chi \) with equal value for all three components breaks \( A_4 \) down to a \( Z_2 \) generated by \( \{1, c, a\} \), and the vev of \( \chi' \) with \( x_2' \) non-zero breaks \( A_4 \) down to a \( Z_2 \) generated by \( \{1, r_2\} \). Here \( a \), \( c \), \( r_2 \) are \( A_4 \) group elements defined in Ref.[3]. Acting on 3, these group elements are represented by

\[
a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
r_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

If in the Higgs potential there are terms directly involve both \( \chi \) and \( \chi' \), it is not possible to have the desired vev structure. This is the so called “sequestering” problem. To separate \( \chi \) from communicating with \( \chi' \) requires additional constraints. This is one of the crucial roles played by SUSY in this model. Without SUSY there is no way to forbid terms of the form \( \chi^{\dagger} \chi' \chi' \) and therefore destroys the desired vev structure in
four dimensional renormalizable theories. With SUSY, potentials are derived from F-terms in the superpotential and D-terms involving gauge interactions. Terms of the type $\chi^I \chi^I \chi^I$ are forbidden if one only allows renormalizable terms in the model.

In the model discussed here the relevant terms in the superpotential consistent with the global symmetries is imposed by

$$W_V = \lambda_\chi^I \chi^I S + \lambda_\chi \chi^I \chi^I S + \mu_{\chi} S^2 + \delta_S S + \lambda_\chi S^3$$

As is well known that soft SUSY breaking terms are needed to construct phenomenologically consistent model, one needs to have these terms here too. Adding all terms which softly break SUSY but keep $A_4 \times Z_4 \times Z_3 \times Z_2$ symmetries, we have

$$V_{soft} = b_1 \chi^I + b_2 \chi^I \chi^I + b_3 S^I + b_4 S^I S$$

This model differs the model in Ref.[4] in that the model constructed here has a simpler Higgs sector, although less predictions for quark mixing.

Using the stationary conditions of the Higgs potential, we obtain

$$x_2 \frac{\partial V}{\partial x_1} - x_1 \frac{\partial V}{\partial x_2} = -2(x_1^2 - x_2^2)(\lambda_\chi^I x_1 x_2)$$

$$+ 6\lambda_\chi^I \chi^I S^3 + c_2 x_3) = 0,$$

$$x_2 \frac{\partial V}{\partial x_3} - x_3 \frac{\partial V}{\partial x_2} = -2(x_3^2 - x_2^2)(\lambda_\chi^I x_3 x_2)$$

$$+ 6\lambda_\chi^I \chi^I S^3 + c_2 x_3) = 0,$$

$$x_2 \frac{\partial V}{\partial x_3} - x_3 \frac{\partial V}{\partial x_2} = -2(x_3^2 - x_2^2)(\lambda_\chi^I x_3 x_2)$$

$$+ 6\lambda_\chi^I \chi^I S^3 + c_2 x_3) = 0,$$

$$x_2 \frac{\partial V}{\partial x_3} - x_3 \frac{\partial V}{\partial x_2} = -2(x_3^2 - x_2^2)(\lambda_\chi^I x_3 x_2)$$

$$+ 6\lambda_\chi^I \chi^I S^3 + c_2 x_3) = 0,$$

$$x_2 \frac{\partial V}{\partial x_3} - x_3 \frac{\partial V}{\partial x_2} = -2(x_3^2 - x_2^2)(\lambda_\chi^I x_3 x_2)$$

$$+ 6\lambda_\chi^I \chi^I S^3 + c_2 x_3) = 0.$$